Today we investigate how to differentiate products of functions and quotients of functions. For example how can we go about differentiating $(x+4)(x^2+5)$ or $\frac{x+4}{x^2+5}$? The rules we will learn today will allow us to take derivatives of products and quotients of functions.

First let's discuss how ***not*** to take the derivative of a product of functions.

Suppose we'd like to take the derivative of $f(x) = x^2(x+2)$ without first expanding the function. Notice that we can take the derivative of x^2 and $x+2$ using derivative rules we've previously discussed. It is natural for us to assume we can take the derivative in the following way.

$$\begin{aligned} f'(x) &\vDash (x^2)'(x+2)', \\ &\vDash (2x)(1+0)', \\ &\vDash 2x. \end{aligned}$$

We use the symbol " $\vDash$" instead of "=" to emphasize that the "rule" we've created is just a hypothesis(and as we'll see incorrect). So according to our psuedo-rule we claim that $f'(x)$ is $2x$. However, in this case, since

$$f(x) = x^2(x+2) = x^3 + 2x^2,$$

we can use our previous derivative rules(the power rule, sum rule, and scalar constant rule) to verify that our answer is correct. According to our previous rules we should get

$$\begin{aligned} f'(x) &= (x^3 + 2x^2)', \\ &= (x^3)' + (2x^2)', \\ &= 3x^2 + 4x. \end{aligned}$$

What we see is that our pseudo-rule doesn't agree with our existing rules. Since we know our existing rules are correct, the rule we created must be **incorrect**. This demonstrates we can't rely solely on intuition.

So why did we go through that example if the rule we created doesn't work? The method we tried above is a very natural way to want to differentiate a product of functions. Unfortunately it turns out to be wrong, and it's important to give a counterexample to avoid making this error in the future.

To summarize, what we found above is that if $f(x)$ and $g(x)$ are differentiable functions then in general

$$\big((f(x)g(x)\big)' \neq f'(x)g'(x).$$

Similarly we might naturally like to think that the derivative of $\dfrac{f(x)}{g(x)}$ is $\dfrac{f'(x)}{g'(x)}$, ***but this is wrong as well***.[1]

If $f(x)$ and $g(x)$ are differentiable functions then in general

$$\left(\frac{f(x)}{g(x)}\right)' \neq \frac{f'(x)}{g'(x)}.$$

[1] The reader is left to come up with their own counterexample.

The following pages contain the rules for differentiating products and quotients of functions.

The Product Rule

If $f(x)$ and $g(x)$ are differentiable functions, then their product is differentiable, and

$$\big(f(x)g(x)\big)' = f'(x)g(x) + f(x)g'(x).$$

Let's consider an example to illustrate the use of this rule.

Example 2 *Differentiate* $Y(x) = x^5(4 + x^2)$.

1. Differentiate $p(x) = \left(x + \dfrac{1}{x}\right)x^7$.

2. Differentiate $f(t) = (t^{-3} + 3)(t^{2.5} - 8t)$.

3. Differentiate $h(x) = (3x^2 + 5x)(54 + x^7)(x + 4)$. *Hint: Use the chain rule twice!*

The Quotient Rule

If $f(x)$ and $g(x)$ are differentiable functions, then their quotient is differentiable, and

$$\left(\frac{f(x)}{g(x)}\right)' = \frac{f'(x)g(x) - f(x)g'(x)}{\big(g(x)\big)^2}.$$

Example 3 *Determine the derivative of* $B(t) = \dfrac{2+t}{\sqrt{t}+40}$.

4. The concentration of urea, in $\mu g/ml$, produced by a kidney cell during an experiment is modeled by the following function, where t is minutes after the experiment began.

$$D(t) = \frac{\sqrt{t}}{t^2+5}.$$

How fast is the concentration changing 30 seconds into the experiment?

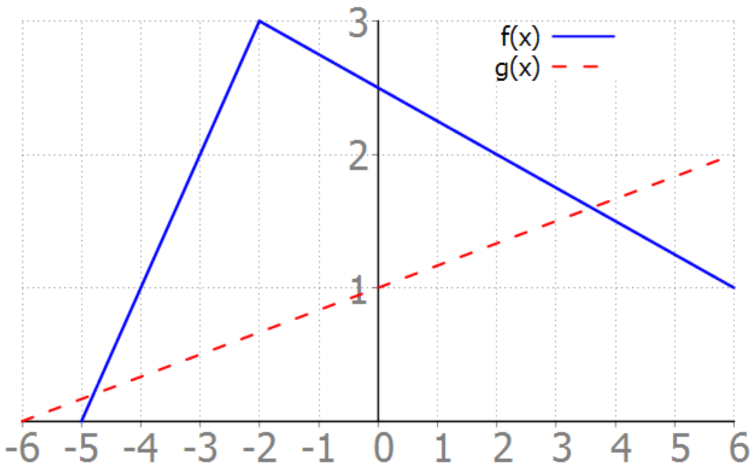
5. Practice the quotient rule by differentiating the following functions. Take the derivatives before you do any simplification.

   (a) $H(x) = \dfrac{x}{x^3 - 2x}$.

   (b) $q(t) = \dfrac{5t^2}{t^3 - 1}$.

   (c) $k(x) = \dfrac{5x^2 + 9x}{17 + x^{-4}}$.

Suppose $h(x) = f(x)g(x)$ and $p(x) = \dfrac{f(x)}{g(x)}$. Use the figure below to answer the following questions. If a derivative does not exist write DNE. *Hint: First write out the definition of the appropriate derivative rule.*



6. (a) Determine $h'(2)$.

(b) Determine $h'(-3)$.

(c) Determine $p'(1)$.

(d) Determine $p'(-2)$.

7. Thoroughbred Bus company finds that its monthly costs for one particular year were given by $C(t) = 10,000 + t^2$ dollars after t months. After t months the company had $P(t) = 1,000 + t^2$ passengers per month. How fast is its cost per passenger changing after 6 months?

8. The Verhulst model for population growth specifies the reproductive rate of an organism as a function of the total population according to the following formula:

$$R(p) = \frac{r}{1 + kp},$$

where p is the total population in thousands of organism, r and k are constants that depend on the organism being studied, and $R(p)$ is the reproduction rate in thousands of organisms per hour. If $k = 0.125$ and $r = 45$, find $R'(4)$. Interpret your answer.

Many of the examples we covered today could have been done simply by using the rules we learned last lecture. However we will learn a few more rules, and in those cases we will have to know the product and quotient rules. For example we cannot differentiate the functions below using only the rules we learned in the last lecture.

$$f(x) = \frac{e^x}{5x^2}, \qquad h(t) = 10t^3 ln(t)$$