Sec 6.1. Eigenvalues and Eigenvectors.

Def. Let A be an $n\times n$ matrix. λ (scalar) is said to be an $\boxed{\text{eigenvalue}}$ of A if there exists a nonzero vector $\bar{x}\in\mathbb{R}^n$ such that $A\bar{x}=\lambda\bar{x}$.
The vector $\bar{x}$ is said to be an $\boxed{\text{eigenvector}}$ of λ.

Remark. The equation $A\bar{x}=\lambda\bar{x}$ is called the eigenvalue equation of A. Eigenvalue and eigenvector are also called characteristic value and characteristic vector.

eg. $A=\begin{bmatrix}4 & -2\\ 1 & 1\end{bmatrix}$ $\quad \bar{x}=\begin{bmatrix}2\\1\end{bmatrix}$

$$A\bar{x}=\begin{bmatrix}6\\3\end{bmatrix}=3\begin{bmatrix}2\\1\end{bmatrix}=3\bar{x}$$

$\lambda=3$ is an eigenvalue of A and $\bar{x}=\begin{bmatrix}2\\1\end{bmatrix}$ is the corresponding eigenvector (of 3).

Theorem λ is an eigenvalue of A if and only if $\boxed{\det(A-\lambda I)=0}$, where $I=\begin{bmatrix}1 & & 0\\ & \ddots & \\ 0 & & 1\end{bmatrix}$ is the $n\times n$ identity matrix.

eg. Find $\boxed{\text{ALL}}$ eigenvalues of $A=\begin{bmatrix}4 & -2\\ 1 & 1\end{bmatrix}$

$$A-\lambda I=\begin{bmatrix}4 & -2\\ 1 & 1\end{bmatrix}-\lambda\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}=\begin{bmatrix}4 & -2\\ 1 & 1\end{bmatrix}-\begin{bmatrix}\lambda & 0\\ 0 & \lambda\end{bmatrix}=\begin{bmatrix}4-\lambda & -2\\ 1 & 1-\lambda\end{bmatrix}$$

$\det(A-\lambda I)=(4-\lambda)(1-\lambda)-(-2)\cdot 1=\lambda^2-5\lambda+4+2=\lambda^2-5\lambda+6=0$

$\Leftrightarrow$ $(\lambda-2)(\lambda-3)=0$ $\Leftrightarrow$ $\lambda=2$ or $\lambda=3$.

The eigenvalues of A are $\lambda_1=2$ and $\lambda_2=3$.

Def. Let A be an $n\times n$ matrix. $p(\lambda)=\det(A-\lambda I)$ is called the **characteristic polynomial** of A, which is an nth degree polynomial in λ.
$p(\lambda)=0$ is called the characteristic equation.

Def. If λ is an eigenvalue of A, then $N(A-\lambda I)$ is called the **eigenspace** corresponding to λ.

Theorem The following statements are equivalent:
(a). λ is an eigenvalue of A.
(b). $(A-\lambda I)\cdot\bar{x}=\bar{0}$ has a nontrivial solution.
(c). $N(A-\lambda I)\neq\{\bar{0}\}$
(d). $A-\lambda I$ is singular
(e). $\det(A-\lambda I)=0$.

eg Find the eigenvalues and the corresponding eigenspace of

$$A=\begin{bmatrix}2 & 6\\ 0 & -1\end{bmatrix}$$

Solution:

$$\det(A-\lambda I)=\begin{vmatrix}2-\lambda & 6\\ 0 & -1-\lambda\end{vmatrix}=(2-\lambda)(-1-\lambda)=0 \Rightarrow \lambda_1=2,\ \lambda_2=-1.$$

$\lambda_1=2$. $A-2I=\begin{bmatrix}0 & 6\\ 0 & -3\end{bmatrix}\longrightarrow\begin{bmatrix}0 & 1\\ 0 & 0\end{bmatrix}$ (reduced row echelon form)

$x_2=0$ (free) x_1 (lead)

$N(A-2I)=\left\{\alpha\cdot\begin{pmatrix}1\\0\end{pmatrix}\ \middle|\ \alpha\in\mathbb{R}\right\}$ is the eigenspace corresponding to $\lambda_1=2$.

$\lambda_2=-1$ $A-(-1)I=\begin{bmatrix}3 & 6\\ 0 & 0\end{bmatrix}\longrightarrow\begin{bmatrix}1 & 2\\ 0 & 0\end{bmatrix}$ $\underset{\text{lead}}{x_1}=-2\underset{\text{free}}{x_2}$

$N(A+I)=\left\{\alpha\cdot\begin{pmatrix}2\\1\end{pmatrix}\ \middle|\ \alpha\in\mathbb{R}\right\}$

Remark. If $A=\begin{bmatrix}\lambda_1 & & *\\ & \lambda_2 & \\ 0 & & \ddots & \\ & & & \lambda_n\end{bmatrix}$ is an upper triangular matrix, then it is easy to check that $p(A)=(\lambda-\lambda_1)(\lambda-\lambda_2)\cdots(\lambda-\lambda_n)$ and $\lambda_1,\cdots,\lambda_n$ are eigenvalues of A.

eg. $A=\begin{bmatrix} 2 & -3 & 1 \\ 1 & -2 & 1 \\ 1 & -3 & 2 \end{bmatrix}$. Find the eigenvalues and the corresponding eigenspaces.

Solution: $\det(A-\lambda I)=\begin{vmatrix} 2-\lambda & -3 & 1 \\ 1 & -2-\lambda & 1 \\ 1 & -3 & 2-\lambda \end{vmatrix} = (2-\lambda)\begin{vmatrix} -2-\lambda & 1 \\ -3 & 2-\lambda \end{vmatrix} + 3\cdot\begin{vmatrix} 1 & 1 \\ 1 & 2-\lambda \end{vmatrix} + \begin{vmatrix} 1 & -2-\lambda \\ 1 & -3 \end{vmatrix}$

$= (2-\lambda)\cdot\left((-2-\lambda)(2-\lambda)+3\right) + 3\cdot(2-\lambda-1) + (-3-(-2-\lambda))$

$= (2-\lambda)(\lambda^2-1) + 3(1-\lambda) + (-1+\lambda)$

$= 2\lambda^2 - 2 - \lambda^3 + \lambda + 3 - 3\lambda - 1 + \lambda$

$= -\lambda^3 + 2\lambda^2 - \lambda = -\lambda(\lambda^2-2\lambda+1) = -\lambda(\lambda-1)^2 = 0$

$\lambda_1 = 0$, $\lambda_2 = \lambda_3 = 1$

$\lambda_1 = 0$, $A - 0\cdot I = \begin{bmatrix} 2 & -3 & 1 \\ 1 & -2 & 1 \\ 1 & -3 & 2 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & -2 & 1 \\ 0 & 1 & -1 \\ 0 & -1 & 1 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$

$\begin{cases} x_1 - x_3 = 0 \\ x_2 - x_3 = 0 \end{cases} \Rightarrow \begin{cases} x_1 = x_3 \\ x_2 = x_3 \end{cases}$. $N(A) = \left\{ \alpha\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \mid \alpha \in \mathbb{R} \right\}$.

$\lambda_2 = 1$
$A - I = \begin{bmatrix} 1 & -3 & 1 \\ 1 & -3 & 1 \\ 1 & -3 & 1 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & -3 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & \end{bmatrix}$ $x_1 - 3x_2 + x_3 = 0$
$x_1 = 3x_2 - x_3$.

$N(A-I) = \left\{ \alpha\cdot\begin{pmatrix} 3 \\ 1 \\ 0 \end{pmatrix} + \beta\begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} \mid \alpha, \beta \in \mathbb{R} \right\}$

Remark. In the above example, we say $\lambda_1 = 0$ has multiplicity one and $\lambda_2 = 1$ has multiplicity 2.

Remark. Not all eigenvalues are real numbers

eg. $A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$, $\det(A-\lambda I) = \begin{vmatrix} -\lambda & -1 \\ 1 & -\lambda \end{vmatrix} = \lambda^2 + 1 = 0$

$\lambda_1 = i$, $\lambda_2 = -i$ are two complex eigenvalues of A.
The corresponding eigenvectors are in $\mathbb{C}^2$ (pair of two complex numbers)

Some important properties about eigenvalues.

- If $\lambda_1, \lambda_2, \dots, \lambda_n$ are eigenvalues of an $n\times n$ matrix A, then

  (1) $\lambda_1 \cdot \lambda_2 \cdot \lambda_3 \cdot \cdots \lambda_n = p(0) = \det(A)$, where $p(\lambda)$ is the characteristic polynomial of A.

  (2). $\lambda_1 + \lambda_2 + \cdots + \lambda_n = a_{11} + a_{22} + \cdots + a_{nn}$
  where $a_{11}, \dots, a_{nn}$ are the diagonal terms of A.

- Remark: For an $n\times n$ matrix A, the sum of its diagonal terms $a_{11} + a_{22} + \cdots + a_{nn}$ is called $\boxed{\text{the trace of } A}$ and is denoted by $tr(A)$.

Def. A matrix B is said to be $\boxed{\text{similar}}$ to a matrix A if there is a nonsingular matrix S such that

$$B = S^{-1} \cdot A \cdot S$$

Theorem. Let A and B be $n\times n$ matrices. If B is similar to A, then A and B have the same

(1) characteristic polynomial $\quad p_A(\lambda) = p_B(\lambda) = p_{S^{-1}AS}(\lambda)$

(2) eigenvalues

(3) determinant $\quad \det(A) = \det(B) = \det(S^{-1}AS)$

(4) trace. $\quad tr(A) = tr(B) = tr(S^{-1}AS)$.

HW 1(a). (c) (f). (h). 2. 3. 4. 14.

9: 6^*, 8^*, 17^*