Sec 5.4. Inner product spaces.

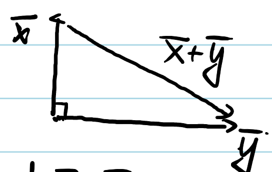
- Recall the definitions of scalar product and length on $\mathbb{R}^n$:
  For $\bar{x}=(x_1,\dots,x_n)^T$, $\bar{y}=(y_1,\dots,y_n)^T \in \mathbb{R}^n$,
  inner (scalar) product: $\langle \bar{x},\bar{y}\rangle = \bar{x}^T\bar{y} = \sum_{i=1}^{n} x_i y_i$
  length : $\|\bar{x}\| = \sqrt{\langle \bar{x},\bar{x}\rangle} = \sqrt{\sum_{i=1}^{n} x_i^2}$
  Sometimes, we also call the above length $\boxed{\text{2-norm of } \bar{x}}$, and write it as $\boxed{\|\bar{x}\|_2}$

  Remark. later we will see there are several other "norms" on $\mathbb{R}^n$. 2-norm is the most commonly used norm on $\mathbb{R}^n$.

- θ between $\bar{x}$ and $\bar{y}$ satisfies: $\cos\theta = \dfrac{\langle \bar{x},\bar{y}\rangle}{\|\bar{x}\|\cdot\|\bar{y}\|}$
- scalar projection of $\bar{x}$ onto $\bar{y}$:
  $$\alpha = \|\bar{x}\|\cdot\cos\theta = \frac{\langle \bar{x},\bar{y}\rangle}{\|\bar{y}\|}$$
- Vector projection of $\bar{x}$ onto $\bar{y}$:
  $$\bar{p} = \alpha\cdot\left(\frac{1}{\|\bar{y}\|}\cdot\bar{y}\right) = \frac{\langle \bar{x},\bar{y}\rangle}{\|\bar{y}\|^2}\cdot\bar{y}$$
- $\bar{x}$ and $\bar{y}$ are orthogonal if $\langle \bar{x},\bar{y}\rangle = 0$ and we write $\bar{x}\perp\bar{y}$.

Theorem (The Pythagorean law)
If $\bar{x}\perp\bar{y}$, then $\|\bar{x}+\bar{y}\|^2 = \|\bar{x}\|^2 + \|\bar{y}\|^2$.

- Geometrically, the Pythagorean law means



eg. Let $\bar{x}=(1,1,0,3)^T$ and $\bar{y}=(1,-1,2,0)^T$
Verify the Pythagorean law on $\bar{x}$, $\bar{y}$ and $\bar{x}+\bar{y}$.

$\langle \bar{x},\bar{y}\rangle = 1-1+0+0=0 \Rightarrow \bar{x}\perp\bar{y}$ ($\bar{x}$ and $\bar{y}$ are orthogonal in $\mathbb{R}^4$)

$\|\bar{x}\|^2 = 1^2+1^2+0^2+3^2 = 11$, $\|\bar{y}\|^2 = 1^2+(-1)^2+2^2+0^2 = 6$

$\bar{x}+\bar{y} = (2,0,2,3)^T$

$\|\bar{x}+\bar{y}\|^2 = 2^2+0^2+2^2+3^2 = 4+0+4+9 = 17$.

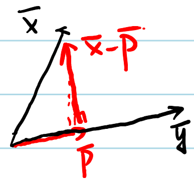
$\|\bar{x}\|^2 + \|\bar{y}\|^2 = 11+6 = 17 = \|\bar{x}+\bar{y}\|^2$.

Theorem. If $\bar{p}$ is the vector projection of $\bar{x}$ onto $\bar{y}$, then.

(1) $\bar{x}-\bar{p} \perp \bar{p}$ and.

(2) $\|\bar{x}-\bar{p}\|^2 + \|\bar{p}\|^2 = \|\bar{x}\|^2$

- Geometricly, it means:



The vector projection $\bar{p}$ is also called orthogonal projection.

eg. (Example in §5.1).

Let $\bar{x}=(2,-5,4)^T$, $\bar{y}=(1,2,-1)^T$

We have computed in §5.1 that the vector projection of $\bar{x}$ onto $\bar{y}$ is:

$\bar{p}=(-2,-4,2)^T$.

We can verify the above theorem using these vectors.

$\bar{x}-\bar{p}=(4,-1,2)^T$.

$\Rightarrow \langle \bar{x}-\bar{p}, \bar{p} \rangle = (4,-1,2)\cdot\begin{pmatrix}-2\\-4\\2\end{pmatrix} = -8+4+4=0.$

$\Rightarrow \bar{x}-\bar{p} \perp \bar{p}.$

$\|\bar{x}-\bar{p}\|^2 = 4^2+(-1)^2+2^2 = 21$, $\|\bar{p}\|^2 = (-2)^2+(-4)^2+2^2 = 24$.

$\|\bar{x}\|^2 = 2^2+(-5)^2+4^2 = 4+25+16=45$.

Indeed, $\|\bar{x}-\bar{p}\|^2 + \|\bar{p}\|^2 = 21+24=45=\|\bar{x}\|^2$.

- Other "norms" on $\mathbb{R}^n$

2-norm: $\|\bar{x}\|_2 = \sum_{i=1}^{n} x_i^2$

1-norm: $\|\bar{x}\|_1 = \sum_{i=1}^{n} |x_i| = |x_1|+\cdots+|x_n|$

∞-norm: $\|\bar{x}\|_\infty = \max_{1\le i\le n} |x_i|$

eg. $\bar{x}=(4,-5,3)^T$. Find $\|\bar{x}\|_2$, $\|\bar{x}\|_1$, $\|\bar{x}\|_\infty$

$\|\bar{x}\|_2 = \sqrt{4^2+(-5)^2+3^2} = \sqrt{50} = 5\sqrt{2}$.

$\|\bar{x}\|_1 = |4|+|-5|+|3| = 12$.

$\|\bar{x}\|_\infty = \max\{|4|, |-5|, |3|\} = 5$.

- 2-norm is induced by the inner product $\|\bar{x}\|_2 = \sqrt{\langle \bar{x},\bar{x}\rangle}$, 1-norm and ∞-norm are not (induced by any inner produce).
- The Pythagorean law holds true for norms induced by inner product, 1-norm and ∞-norm do not satisfy the Pythagorean law.

- Consider the previous example

$\bar{x}=(1,1,0,3)^T$, $\bar{y}=(1,-1,2,0)^T$

$\bar{x}+\bar{y}=(2,0,2,3)$

$\|\bar{x}\|_\infty=3$, $\|\bar{y}\|_\infty=2$, $\|\bar{x}+\bar{y}\|_\infty=3$

$3^2+2^2\neq 3^2$

$\|\bar{x}\|_1=1+1+3=5$, $\|\bar{y}\|_1=1+1+2=4$, $\|\bar{x}+\bar{y}\|_1=2+2+3=7$

$5^2+4^2\neq 7^2$

- Let V be a vector space

An inner product $\langle\cdot,\cdot\rangle$ is a map from $V\times V$ to $\mathbb{R}$,

A norm $\|\cdot\|$ is a map from V to $\mathbb{R}$.

There are certain conditions on $\langle\cdot,\cdot\rangle$ and $\|\cdot\|$ which generalize those properties of scalar product and length on $\mathbb{R}^n$.

- Conditions on inner product.

I. $\langle x,x\rangle\geq 0$ with equality iff $x=0$.

II. $\langle x,y\rangle=\langle y,x\rangle$

III. $\langle \alpha x+\beta y, z\rangle=\alpha\langle x,z\rangle+\beta\langle y,z\rangle$

- Conditions on norm

I $\|x\|\geq 0$ with equality iff $x=0$.

II. $\|\alpha x\|=|\alpha|\cdot\|x\|$

III. $\|x+y\|\leq\|x\|+\|y\|$

- A vector space V equipped with an inner product is called an inner product space
- A vector space V equipped with a norm is called a norm space.

eg. $\mathbb{R}^n$ with the usual scalar product (as inner product) is an inner product space

The proof of triangle inequality follows from the Cauchy-Schwarz inequality

$$|\langle\bar{x},\bar{y}\rangle|\leq\|\bar{x}\|\cdot\|\bar{y}\|.$$