Sec 5.3 Least Square Problems

· Given an $m\times n$ matrix A and a vector $\bar{b}\in\mathbb{R}^m$
the system $A\bar{x}=\bar{b}$ may be inconsistent
ie. there may not be a $\bar{x}\in\mathbb{R}^n$ such that
$$A\bar{x}-\bar{b}=\bar{0}$$

Question: How to find a $\bar{x}\in\mathbb{R}^n$ such that
$A\bar{x}$ is closetest to $\bar{b}$. ?

e.g. $A=\begin{pmatrix}1&1\\-2&3\\2&-1\end{pmatrix}$, $\bar{b}=\begin{pmatrix}3\\1\\2\end{pmatrix}$ Consider $A\bar{x}=\bar{b}$

$$\left(\begin{array}{cc:c}1&1&3\\-2&3&1\\2&-1&2\end{array}\right)\to\left(\begin{array}{cc:c}1&1&3\\0&5&7\\0&2&3\end{array}\right)\to\left(\begin{array}{cc:c}1&1&3\\0&1&\frac{7}{5}\\0&0&3-\frac{14}{5}\end{array}\right)$$
inconsistent.

Thm. If A is an $m\times n$ matrix of rank n, then
the linear system $A^TA\bar{x}=A^T\bar{b}$ has a solution $\hat{x}$
such that
$\|A\bar{x}-\bar{b}\|$ obtains its minimum at $\hat{x}$.

$\hat{x}$ is called the least squares solution of $A\bar{x}=\bar{b}$.

$$A^TA=\begin{bmatrix}1&-2&2\\1&3&-1\end{bmatrix}\cdot\begin{bmatrix}1&1\\-2&3\\2&-1\end{bmatrix}=\begin{bmatrix}9&-7\\-7&11\end{bmatrix}$$

$$A^T\bar{b}=\begin{bmatrix}1&-2&2\\1&3&-1\end{bmatrix}\cdot\begin{bmatrix}3\\1\\2\end{bmatrix}=\begin{bmatrix}5\\4\end{bmatrix}.$$

We want to consider the new system. $A^TA\bar{x} = A^T\bar{b}$

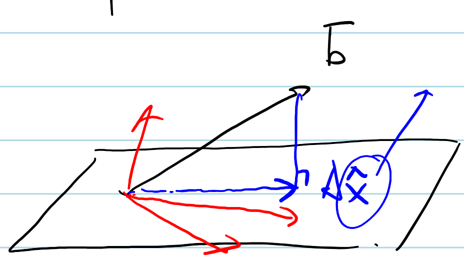
i.e., $\begin{bmatrix} 9 & -7 \\ -7 & 11 \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 5 \\ 4 \end{bmatrix}$

$$\left[\begin{array}{cc|c} 9 & -7 & 5 \\ -7 & 11 & 4 \end{array}\right] \longrightarrow \left[\begin{array}{cc|c} 9 & -7 & 5 \\ 0 & \frac{50}{9} & \frac{71}{9} \end{array}\right] \; \frac{7}{9}$$

$$11 - \frac{49}{9} \qquad 4 + \frac{35}{9}$$

$$\begin{cases} 9x_1 - 7x_2 = 5 \\ \frac{50}{9} x_2 = \frac{71}{9} \end{cases} \Rightarrow \begin{cases} x_1 = \frac{83}{50} \\ x_2 = \frac{71}{50} \end{cases}$$

Geometric explanation



least squares solution

$S = \mathrm{span}\left(\begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix}, \begin{pmatrix} 1 \\ 3 \\ -1 \end{pmatrix}\right)$

$A\bar{x} \in S$

One can prove that

$A\hat{x}$ is the projection of $\bar{b}$ onto the plane.

$\hat{x} = (A^TA)^{-1} \cdot A^T\bar{b}$

$A\hat{x} = \underbrace{A \cdot (A^TA)^{-1} \cdot A^T}\,\bar{b}$ is the projection of $\bar{b}$ onto the plane

The matrix $P = A(A^TA)^{-1}A^T$ is called the projection matrix.

P satisfies

$P^2 = P$ and $P^T = P$

Proof. $P^2 = P \cdot P = A \underbrace{(A^T A)^{-1} \cdot A^T \cdot A}_{} (A^T A)^{-1} A^T$

$= A \cdot I \cdot (A^T A)^{-1} A^T$

$= A \cdot (A^T A)^{-1} A^T = P$

$P^T = \left(\underbrace{A}_{} \underbrace{(A^T A)^{-1}}_{} \cdot \underbrace{A^T}_{} \right)^T$

$= (A^T)^T \cdot \left((A^T A)^{-1} \right)^T \cdot A^T$

$= A \cdot \left((A^T A)^T \right)^{-1} \cdot A^T$ we use the fact that $(B^{-1})^T = (B^T)^{-1}$ for any nonsingular B.

$= A (A^T \cdot A)^{-1} \cdot A^T$

$= P$.