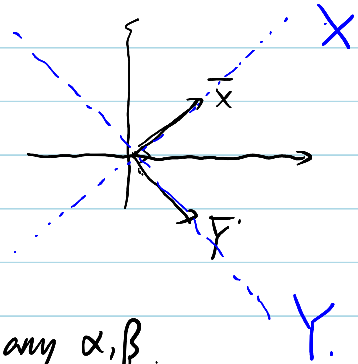
Sec 3.2. Orthogonal subspaces.

Def. Two subspaces X and Y of $\mathbb{R}^n$ are said to be orthogonal if $\bar{x}^T \cdot \bar{y} = 0$ for all $\bar{x} \in X$ and $\bar{y} \in Y$. We write $X \perp Y$.

e.g. $\bar{x} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ and $\bar{y} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$.



$\bar{x}^T \cdot \bar{y} = 0$

$X = \mathrm{span}\left(\begin{pmatrix} 1 \\ 1 \end{pmatrix}\right)$, $Y = \mathrm{span}\left(\begin{pmatrix} 1 \\ -1 \end{pmatrix}\right)$

$\bar{x}^T \cdot \bar{y} = 0 \Rightarrow (\alpha \bar{x})^T \cdot (\beta \cdot \bar{y}) = 0$ for any α, β.

$X \perp Y$.

Def. $Y^\perp = \{ \bar{x} \in \mathbb{R}^n \mid \bar{x}^T \cdot \bar{y} = 0 \text{ for all } \bar{y} \in Y \}$ is called the orthogonal complement of Y in $\mathbb{R}^n$.

$Y^\perp$ is the collection of all vectors that are orthogonal to every vector in Y.

e.g. $X = Y^\perp$ and $Y = X^\perp$ in the first example.

e.g. Let S be the subspace of $\mathbb{R}^3$ spanned by $\bar{x} = (1, -1, 2)^T$

Find $S^\perp$.

$S = \mathrm{span}(\bar{x}) = \{ \alpha \cdot (1, -1, 2)^T \mid \alpha \in \mathbb{R} \}$

If $\bar{y}^T \cdot \bar{x} = 0$, then $\bar{y}^T \cdot (\alpha \cdot \bar{x}) = 0$

Therefore, it is enough to find all vectors that are orthogonal to $(1, -1, 2)^T$.

$$\bar{y}^T \cdot \bar{x} = 0 \iff (y_1, y_2, y_3) \cdot \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix} = y_1 - y_2 + 2y_3 = 0$$

$\Rightarrow y_1 = y_2 - 2y_3 = \alpha - 2\beta$, $y_2 = \alpha$, $y_3 = \beta$.

$$\begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = \begin{pmatrix} \alpha - 2\beta \\ \alpha \\ \beta \end{pmatrix} = \alpha \cdot \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + \beta \cdot \begin{pmatrix} -2 \\ 0 \\ 1 \end{pmatrix}, \quad S^\perp = \left\{ \alpha \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + \beta \begin{pmatrix} -2 \\ 0 \\ 1 \end{pmatrix} \,\middle|\, \alpha, \beta \in \mathbb{R} \right\}.$$

e.g. let $S = \operatorname{span}\left(\begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix} \right)$

$\qquad\qquad\qquad\qquad\quad \begin{matrix} \| & \| \\ v_1 & v_2 \end{matrix}$.

Find $S^\perp$.

Let $A = \begin{pmatrix} v_1^\perp \\ v_2^\perp \end{pmatrix} = \begin{pmatrix} 1 & 2 & 1 \\ 1 & -1 & 2 \end{pmatrix}$. Then $S^\perp = N(A)$

If $\bar{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$ satisfies $v_1^\perp \cdot \bar{x} = 0$ and $v_2^\perp \cdot \bar{x} = 0$.

then $(c_1 \cdot v_1 + c_2 v_2)^\perp \cdot \bar{x} = 0$. for all c_1, c_2.

$$\begin{cases} v_1^\perp \bar{x} = 0 \\ v_2^\perp \bar{x} = 0 \end{cases} \iff A \cdot \bar{x} = 0. \iff \bar{x} \in N(A)$$

$$\left[\begin{array}{ccc:c} 1 & 2 & 1 & 0 \\ 1 & -1 & 2 & 0 \end{array}\right] \longrightarrow \left[\begin{array}{ccc:c} 1 & 2 & 1 & 0 \\ 0 & -3 & 1 & 0 \end{array}\right]$$

$$\longrightarrow \left[\begin{array}{ccc:c} 1 & 2 & 1 & 0 \\ 0 & 1 & -\frac{1}{3} & 0 \end{array}\right] \rightarrow \left[\begin{array}{ccc:c} 1 & 0 & \frac{5}{3} & 0 \\ 0 & 1 & -\frac{1}{3} & 0 \end{array}\right]$$

$$\begin{cases} x_1 = -\frac{5}{3} x_3, \\ x_2 = +\frac{1}{3} x_3, \end{cases} \quad x_3 = \alpha, \quad \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} -\frac{5}{3}\alpha \\ \frac{1}{3}\alpha \\ \alpha \end{pmatrix} = \alpha \cdot \begin{pmatrix} -\frac{5}{3} \\ \frac{1}{3} \\ 1 \end{pmatrix}.$$

$$S^\perp = N(A) = \operatorname{span}\left(\alpha \cdot \begin{pmatrix} -\frac{5}{3} \\ \frac{1}{3} \\ 1 \end{pmatrix} \right)$$