Chapter 5. Orthogonality.

Sec 5.1 The scalar product in $\mathbb{R}^n$

Def. Given two (column) vectors $\bar{x}, \bar{y} \in \mathbb{R}^n$,
the scalar product of $\bar{x}$, and $\bar{y}$ is defined to be

$$\bar{x}^T \cdot \bar{y} = \sum_{i=1}^{n} x_i y_i = x_1 y_1 + \cdots + x_n y_n \quad \text{for } \bar{x} = (x_1, \cdots, x_n)^T,\ \bar{y} = (y_1, \cdots, y_n)^T$$

Remark $\bar{x}^T$ is a "$1 \times n$" matrix and $\bar{y}$ is an "$n \times 1$" matrix. $\bar{x}^T \cdot \bar{y}$ is the usual matrix product.

Remark. The scalar product of two $\mathbb{R}^n$ vectors is also called dot product.

In section 5.4, we will introduce another concept/notation called INNER PRODUCT for the scalar product and write

$$\langle \bar{x}, \bar{y} \rangle = \bar{x}^T \cdot \bar{y} = \sum_{i=1}^{n} x_i y_i .$$

Here, $\langle \cdot, \cdot \rangle$ serves as a "two-variable function" from $\mathbb{R}^n \times \mathbb{R}^n$ to $\mathbb{R}$, mapping each pair of $\bar{x}, \bar{y}$ to a real number $\langle \bar{x}, \bar{y} \rangle$.

e.g. If $\bar{x} = \begin{bmatrix} 3 \\ -2 \\ 1 \end{bmatrix}$, $\bar{y} = \begin{bmatrix} 4 \\ 3 \\ 2 \end{bmatrix}$, then

$$\bar{x}^T \cdot \bar{y} = 3 \cdot 4 + (-2) \cdot 3 + 1 \cdot 2 = 8.$$

We can also write $\langle \bar{x}, \bar{y} \rangle = 3 \cdot 4 + (-2) \cdot 3 + 1 \cdot 2 = 8$.

- We can use scalar product to define "length", "distance" and "angle" in $\mathbb{R}^n$.

Def. Let $\bar{x} = (x_1, \cdots, x_n)^T$ and $\bar{y} = (y_1, \cdots, y_n)^T$ be vectors in $\mathbb{R}^n$. Then

(1) the length of $\bar{x}$ is defined to be

$$\|\bar{x}\| = \sqrt{\bar{x}^T \cdot \bar{x}} = \sqrt{x_1^2 + x_2^2 + \cdots + x_n^2}$$

(2) the distance between $\bar{x}$, and $\bar{y}$ is defined to be

$$\|\bar{x} - \bar{y}\| = \sqrt{(x_1 - y_1)^2 + \cdots + (x_n - y_n)^2}$$

(3) the angle between $\bar{x}$ and $\bar{y}$, θ, satisfies

$$\bar{x}^T \cdot \bar{y} = \|x\| \cdot \|y\| \cdot \cos\theta \iff \cos\theta = \frac{\bar{x}^T \cdot \bar{y}}{\|\bar{x}\| \cdot \|\bar{y}\|} \quad \text{for } \bar{x} \neq \bar{0}, \bar{y} \neq \bar{0}, \quad 0 \le \theta \le \pi.$$

Remark (1) $\|\bar{0}\| = 0$ (zero vector has length zero) and $\|\bar{x}\| > 0$ for all $\bar{x} \neq \bar{0}$.
(2) $\|\bar{x} - \bar{y}\| = \|\bar{y} - \bar{x}\|$ for all $\bar{x}, \bar{y} \in \mathbb{R}^n$
(3) $\|\alpha \bar{x}\| = |\alpha| \cdot \|\bar{x}\|$ for all $\bar{x} \in \mathbb{R}^n$ and scalar $\alpha \in \mathbb{R}$.

e.g. Let $\bar{x} = (3,4)^T$, $y = (-1,7)^T$

$\|\bar{x}\| = \sqrt{3^2+4^2} = 5$, $\|\bar{y}\| = \sqrt{(-1)^2+7^2} = \sqrt{50} = \sqrt{25\cdot 2} = 5\sqrt{2}$

$\|\bar{x}-\bar{y}\| = \sqrt{(3-(-1))^2+(4-7)^2} = \sqrt{4^2+3^2} = 5.$

$\bar{x}^T\cdot\bar{y} = 3\cdot(-1)+4\cdot 7 = -3+28 = 25.$

$\cos\theta = \frac{\bar{x}^T\cdot\bar{y}}{\|\bar{x}\|\cdot\|\bar{y}\|} = \frac{25}{5\cdot 5\sqrt{2}} = \frac{1}{\sqrt{2}}. \Rightarrow \theta = \frac{\pi}{4}$

Remark. The angle θ is taken values in $[0,\pi]$

Def. Two vectors $\bar{x}, \bar{y}$ in $\mathbb{R}^n$ are said to be orthogonal if $\bar{x}^T\cdot\bar{y} = 0$.

e.g. (1) For any $\bar{x}\in\mathbb{R}^n$, $\bar{x}^T\cdot\bar{0} = 0$. Therefore, the zero vector $\bar{0}$ is orthogonal to any $\bar{x}$.

(2) $\begin{bmatrix}3\\2\end{bmatrix}$ and $\begin{bmatrix}-4\\6\end{bmatrix}$ are orthogonal in $\mathbb{R}^2$ since $(3\ \ 2)\begin{bmatrix}-4\\6\end{bmatrix} = -12+12 = 0$

(3) $\begin{bmatrix}2\\-3\\1\end{bmatrix}$ and $\begin{bmatrix}1\\1\\1\end{bmatrix}$ are orthogonal in $\mathbb{R}^3$ since $2\cdot 1+(-3)\cdot 1+1\cdot 1 = 0$

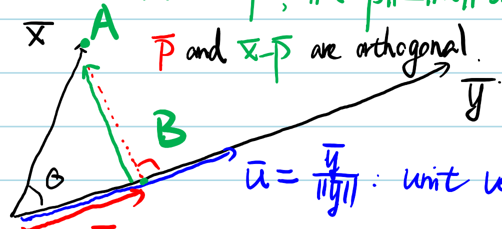
Def. Let $\bar{x}, \bar{y}$ be two vectors in $\mathbb{R}^n$. Let θ be the angle between $\bar{x}$ and $\bar{y}$.
Then the scalar $\alpha = \|\bar{x}\|\cdot\cos\theta$ is called the scalar projection of $\bar{x}$ onto $\bar{y}$.
The vector $\bar{u} = \frac{1}{\|\bar{y}\|}\cdot\bar{y}$ is called the unit vector in the direction of $\bar{y}$.
The vector $\bar{p} = \alpha\cdot\bar{u}$ is called the vector projection of $\bar{x}$ onto $\bar{y}$.

- Formulas: $\alpha = \frac{\bar{x}^T\cdot\bar{y}}{\|\bar{y}\|}$.

$$\bar{p} = \frac{\bar{x}^T\cdot\bar{y}}{\|\bar{y}\|^2}\cdot\bar{y}$$

$\overrightarrow{BA} = \bar{x}-\bar{p}$, $\|\bar{x}-\bar{p}\| = \|\bar{x}\|\cdot\sin\theta$

$\bar{p}$ and $\bar{x}-\bar{p}$ are orthogonal.



$\bar{u} = \frac{\bar{y}}{\|\bar{y}\|}$: unit vector

Let A, B be the endpoints of $\bar{x}$ and $\bar{p}$
Then B is the point that is closest to A.
The distance between A and B is
$\|\bar{x}-\bar{p}\| = \|\bar{x}\|\cdot\sin\theta$.

$\bar{p}$ is the projection of $\bar{x}$ onto $\bar{y}$, and $\bar{p}$ has length $|\alpha|$.
since $\|\bar{p}\| = \left|\frac{\bar{x}^T\cdot\bar{y}}{\|\bar{y}\|^2}\right|\|\bar{y}\| = \frac{|\bar{x}^T\cdot\bar{y}|}{\|\bar{y}\|} = |\alpha|$.

$\bar{u}$ has length 1 since
$\|\bar{u}\| = \left\|\frac{\bar{y}}{\|\bar{y}\|}\right\| = \frac{\|\bar{y}\|}{\|\bar{y}\|} = 1$.

eg. $\bar{x} = \begin{pmatrix} 2 \\ -5 \\ 4 \end{pmatrix}$, $\bar{y} = \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}$

$\langle \bar{x}, \bar{y} \rangle = 2 \cdot 1 + (-5) \cdot 2 + 4 \cdot (-1) = 2 - 10 - 4 = -12.$

$\|\bar{x}\| = \sqrt{2^2 + (-5)^2 + 4^2} = \sqrt{4 + 25 + 16} = \sqrt{45} = 3\sqrt{5}$

$\|\bar{y}\| = \sqrt{1^2 + 2^2 + (-1)^2} = \sqrt{1 + 4 + 1} = \sqrt{6}$

projection of $\bar{x}$ onto $\bar{y}$.

scalar projection: $\alpha = \frac{\bar{x}^T \cdot \bar{y}}{\|\bar{y}\|} = \frac{-12}{\sqrt{6}}$ $\left(= \frac{-12\sqrt{6}}{6} = -2\sqrt{6} \right)$

unit vector in $\bar{y}$: $\bar{u} = \frac{1}{\|\bar{y}\|} \cdot \bar{y} = \frac{1}{\sqrt{6}} (1, 2, -1)^T$

vector projection: $\bar{p} = \alpha \cdot \bar{u} = -2\sqrt{6} \cdot \frac{1}{\sqrt{6}} \cdot (1, 2, -1)^T = (-2, -4, 2)^T$

(or $\bar{p} = \frac{\bar{x}^T \cdot \bar{y}}{\|\bar{y}\|^2} \cdot \bar{y} = \frac{-12}{6} (1, 2, -1)^T$)

April 12. HW. Sec 5.1. 1. 2. (a). (b). 3. (a). (c). 17.