Sec 3.5 Change of Basis

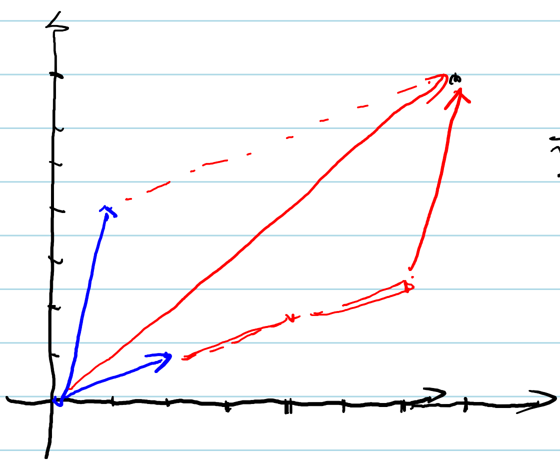
Changing Coordinates in $\mathbb{R}^n$.

$\mathbb{R}^2$:

Standard basis $\bar{e}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, $\bar{e}_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

$\bar{v}_1 = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$, $\bar{v}_2 = \begin{pmatrix} 1 \\ 4 \end{pmatrix}$ form a basis for $\mathbb{R}^2$.

We write $\{\bar{v}_1, \bar{v}_2\}$ as $[\bar{v}_1, \bar{v}_2]$ to indicate the order



$\bar{x} = \begin{pmatrix} 7 \\ 7 \end{pmatrix} = 3 \cdot \bar{v}_1 + 1 \cdot \bar{v}_2$

- The coordinate vector of $\bar{x}$ with respect to $[\bar{v}_1, \bar{v}_2]$ is $\begin{pmatrix} 3 \\ 1 \end{pmatrix}$
- The coordinate vector of $\bar{x}$ with respect to $[\bar{v}_2, \bar{v}_1]$ is $\begin{pmatrix} 1 \\ 3 \end{pmatrix}$.
- In general, the question is to find c_1, c_2 such that

$$\bar{x} = c_1 \bar{v}_1 + c_2 \cdot \bar{v}_2 = (\bar{v}_1, \bar{v}_2) \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$$

$\{v_1, v_2\}$, $\{u_1, u_2\}$ are two sets of basis for $\mathbb{R}^2$.
$\{e_1, e_2\}$ is the standard basis.

$V = (v_1, v_2)$ is the 2×2 matrix formulated by v_1, v_2.

- V is called the transition matrix from $\{v_1, v_2\}$ to $\{e_1, e_2\}$.
- Given any vector $\bar{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$, the coordinates of $\bar{x}$ with respect to $\{v_1, v_2\}$ are given by

$$V^{-1} \cdot \bar{x}$$

- V^{-1} is called the transition matrix from $\{e_1, e_2\}$ to $\{v_1, v_2\}$

$U = (u_1, u_2)$ is the 2×2 matrix formulated by u_1, u_2

$U^{-1} V$ is the transition matrix from $\{v_1, v_2\}$ to $\{u_1, u_2\}$

$$\begin{tikzcd}
{[v_1, v_2]} \arrow[r, shift left, "V"] \arrow[dr, shift right, "U^{-1}V"'] & {[e_1, e_2]} \arrow[l, shift left, "V^{-1}"] \arrow[d, shift left, "U^{-1}"] \\
 & {[u_1, u_2]} \arrow[u, shift left, "U"] \arrow[ul, shift right, "V^{-1} \cdot U"']
\end{tikzcd}$$

eg. $\bar{v}_1=\begin{pmatrix}5\\2\end{pmatrix}$, $\bar{v}_2=\begin{pmatrix}7\\3\end{pmatrix}$, $u_1=\begin{pmatrix}3\\2\end{pmatrix}$, $u_2=\begin{pmatrix}1\\1\end{pmatrix}$

(1) Let $\bar{x}=\begin{pmatrix}7\\4\end{pmatrix}$. Find the coordinates of $\bar{x}$ with respect to $[u_1, u_2]$.

(2) Find the transition matrix from $\{v_1, v_2\}$ to $\{u_1, u_2\}$

Solution:

(1) $U=\begin{bmatrix}3&1\\2&1\end{bmatrix}$.

New coordinates: $U^{-1}\bar{x}=\begin{bmatrix}1&-1\\-2&3\end{bmatrix}\begin{bmatrix}7\\4\end{bmatrix}=\begin{bmatrix}3\\-2\end{bmatrix}$

(2) $V=\begin{bmatrix}5&7\\2&3\end{bmatrix}$. Transition matrix from $[v_1, v_2]$ to $\{u_1, u_2]$

$$T=U^{-1}V=\begin{bmatrix}1&-1\\-2&3\end{bmatrix}\cdot\begin{bmatrix}5&7\\2&3\end{bmatrix}=\begin{bmatrix}3&4\\-4&-5\end{bmatrix}$$

Remark. Transition matrix from $[u_1, u_2]$ to $[v_1, v_2]$:

$$T^{-1}=V^{-1}U=\frac{1}{-15+16}\begin{bmatrix}-5&-4\\4&3\end{bmatrix}=\begin{bmatrix}-5&-4\\4&3\end{bmatrix}$$

The coordinates of $\bar{x}$ with respect to $[v_1, v_2]$ are:

$$\begin{bmatrix}-5&-4\\4&3\end{bmatrix}\cdot\begin{bmatrix}3\\-2\end{bmatrix}=(V^{-1}U)\cdot(U^{-1}\bar{x})=V^{-1}\bar{x}.$$

Hw. 1.(a). 3(b).

Sec 3.6. Rank and nullity.

Def. For a $m \times n$ matrix A, the $\boxed{\text{rank}}$ of A, denoted $\text{rank}(A)$, is the number of non-zero rows in the row echelon form of A.

Def. Let $N(A)$ be the null space of A, i.e., the solution space of $A\bar{x} = \bar{0}$.

The dimension of $N(A)$ is called the nullity of A

Theorem. For a $m \times n$ matrix A,

The rank of A + The nullity of $A = n$.

Remark. Consider the linear system $A\bar{x} = \bar{0}$.

Rank of A = the number of the lead variables

Nullity of A = the number of the free variables

Theorem. For a $n \times n$ matrix A,

A is nonsingular if and only if

$\text{rank}(A) = n$. (full rank)

eg. Let $A = \begin{bmatrix} 1 & 2 & -1 & 1 \\ 2 & 4 & -3 & 0 \\ 1 & 2 & 1 & 5 \end{bmatrix}$

Find the rank and the nullity of A.

$$A \to \begin{bmatrix} 1 & 2 & -1 & 1 \\ 0 & 0 & -1 & -2 \\ 0 & 0 & 2 & 4 \end{bmatrix}$$

$$\to \begin{bmatrix} 1 & 2 & -1 & 1 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$ Raw echelon form.

$\text{Rank}(A) = 2$. $\text{Nullity} = 4 - 2 = 2$.

Remark: To find $N(A)$, we need move forward to the reduced raw echelon form $\begin{bmatrix} 1 & 2 & 0 & 3 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$.

Then. $\begin{cases} x_1 = -2x_2 - 3x_4 \\ x_3 = \quad -2x_4. \end{cases}$

$x_2 = \alpha$, $x_4 = \beta$.

$N(A) = \left\{ \alpha \cdot \begin{pmatrix} -2 \\ 1 \\ 0 \\ 0 \end{pmatrix} + \beta \cdot \begin{pmatrix} -3 \\ 0 \\ -2 \\ 1 \end{pmatrix} \;\middle|\; \alpha, \beta \in \mathbb{R} \right\}$ has dimension 2.