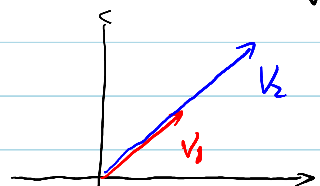
Sec 3.3. Linear Independence.

e.g. Consider the vector space $\mathbb{R}^2$.

Question: could the two vectors $V_1=\begin{pmatrix}1\\1\end{pmatrix}$, $V_2=\begin{pmatrix}2\\2\end{pmatrix}$ span $\mathbb{R}^2$?

Answer: No.

Geometric Interpretation:



V_1, V_2 are in the same direction. Any linear combination of V_1, V_2, $C_1\cdot V_1+C_2\cdot V_2$, is still in this direction. Therefore, vectors not in this direction cannot be expressed as a linear combination of V_1, V_2.

It is easy to see that $2V_1=V_2 \iff 2\cdot V_1 - V_2 = 0$

We generalize this relation to any abstract vector space V.

Def. The vectors $V_1, \cdots, V_n$ in a vector space V are called $\boxed{\text{linearly dependent}}$ if there exists $C_1, \cdots, C_n$ (scalars) not all zero, such that

$$C_1\cdot V_1 + C_2\cdot V_2 + \cdots + C_n\cdot V_n = 0$$

e.g. In $\mathbb{R}^3$, $V_1=\begin{pmatrix}1\\1\\1\end{pmatrix}$, $V_2=\begin{pmatrix}1\\-1\\3\end{pmatrix}$, $V_3=\begin{pmatrix}1\\0\\2\end{pmatrix}$ are linearly dependent since

$$V_1+V_2-2V_3 = \begin{pmatrix}2\\0\\4\end{pmatrix} - 2\cdot\begin{pmatrix}1\\0\\2\end{pmatrix} = \begin{pmatrix}0\\0\\0\end{pmatrix}.$$

Def. If $V_1, \cdots, V_n$ are NOT linearly dependent, they are called $\boxed{\text{linearly independent}}$

Equivalently, $V_1, \cdots, V_n$ are linearly independent if

$$C_1\cdot V_1 + C_2\cdot V_2 + \cdots + C_n V_n \text{ implies } C_1=C_2=\cdots=C_n=0.$$

e.g. In P_3 (polynomial space of degree less than 3),

$p_1(x)=1$, $p_2(x)=x$, $p_3(x)=x^2$ are linearly independent since:

if $C_1\cdot p_1(x)+C_2\, p_2(x)+C_3\, p_3(x) = C_1+C_2 x + C_3 x^2 = 0$ (for all x).

then $C_1=C_2=C_3=0$.

Theorem: If some vectors in $v_1, \dots, v_n$ are linearly dependent, then $v_1, \dots, v_n$ are linearly dependent

If $v_1, \dots, v_n$ are linearly dependent, then any of them are linearly independent.

Theorem: In $\mathbb{R}^n$, given n (column) vectors $\bar{x}_1, \bar{x}_2, \dots, \bar{x}_n$, consider the matrix $X = (\bar{x}_1, \bar{x}_2, \dots, \bar{x}_n)$ ($n \times n$ matrix consisting $\bar{x}_1, \dots, \bar{x}_n$ as its column vectors)

$\bar{x}_1, \dots, \bar{x}_n$ are linearly dependent if and only if $\det X = 0$.

...... (linearly independent) .... ($\det X \neq 0$)

e.g. $\begin{pmatrix}1\\1\end{pmatrix}, \begin{pmatrix}1\\2\end{pmatrix}$ are linearly $\boxed{\text{independent}}$ since $\begin{vmatrix}1 & 1\\1 & 2\end{vmatrix} = 1\cdot 2 - 1\cdot 1 \neq 0$

(two vectors in $\mathbb{R}^2$)

e.g. $\begin{pmatrix}4\\2\\3\end{pmatrix}, \begin{pmatrix}2\\3\\1\end{pmatrix}, \begin{pmatrix}2\\-5\\3\end{pmatrix}$ are linearly $\boxed{\text{dependent}}$ in $\mathbb{R}^3$ since

$$\begin{vmatrix}4 & 2 & 2\\2 & 3 & -5\\3 & 1 & 3\end{vmatrix} = 4\cdot\begin{vmatrix}3 & -5\\1 & 3\end{vmatrix} - 2\cdot\begin{vmatrix}2 & -5\\3 & 3\end{vmatrix} + 2\cdot\begin{vmatrix}2 & 3\\3 & 1\end{vmatrix}$$

$$= 4(9+5) - 2\cdot(6+15) + 2\cdot(2-9)$$

$$= 4\cdot 14 - 2\cdot 21 - 2\cdot 7 = 56 - 42 - 14 = 0$$

HW 1 (a) (b) (c) 2. (a) (c). (e)

Mon. 1. (c). 2. (e). 8. (a) (b) . 9. 10*.

- In general, checking whether given vectors are linearly independent is equivalent to check that whether an associated linear system has non-trivial solutions or not.

e.g. $\begin{pmatrix}1\\0\\0\end{pmatrix}, \begin{pmatrix}0\\1\\0\end{pmatrix}, \begin{pmatrix}1\\0\\1\end{pmatrix}, \begin{pmatrix}1\\2\\3\end{pmatrix}$ are linearly independent or not in $\mathbb{R}^3$?

Set: $c_1 \cdot \begin{bmatrix}1\\0\\0\end{bmatrix} + c_2 \cdot \begin{bmatrix}0\\1\\0\end{bmatrix} + c_3 \cdot \begin{bmatrix}1\\0\\1\end{bmatrix} + c_4 \cdot \begin{bmatrix}1\\2\\3\end{bmatrix} = \begin{bmatrix}0\\0\\0\end{bmatrix}$

$\Leftrightarrow \begin{bmatrix}1 & 0 & 1 & 1\\0 & 1 & 0 & 2\\0 & 0 & 1 & 3\end{bmatrix}\begin{bmatrix}c_1\\c_2\\c_3\\c_4\end{bmatrix} = \begin{bmatrix}0\\0\\0\end{bmatrix}$ has nontrivial solution or not.

$\Leftrightarrow \left[\begin{array}{cccc|c}1 & 0 & 1 & 1 & 0\\0 & 1 & 0 & 2 & 0\\0 & 0 & 1 & 3 & 0\end{array}\right]$ (The coefficient matrix is in raw echolon form already).

$\longrightarrow \left[\begin{array}{cccc|c}1 & 0 & 0 & -2 & 0\\0 & 1 & 0 & 2 & 0\\0 & 0 & 1 & 3 & 0\end{array}\right]$ (Reduced raw echolen form)

Lead variables c_1, c_2, c_3

Free variable: c_4.

$$\begin{cases} c_1 - 2c_4 = 0 \\ c_2 + 2c_4 = 0 \\ c_3 + 3c_4 = 0 \end{cases} \Rightarrow \begin{cases} c_1 = 2c_4 \\ c_2 = -2c_4 \\ c_3 = -3c_4 \end{cases}$$

Take $c_4 = 1$, then $c_1 = 2$, $c_2 = -2$, $c_3 = -3$.

Therefore, $(2, -2, -3, 1)$ is a nontrivial solution to the system and the four vectors are linearly dependent.