§3.2 Subspaces

Def. Given a vector space V with the two operations $+$, $\cdot$,
a non-empty subset S of V is called a subspace of V if
the elements in S are closed under the two operations, i.e.,

(i) $\alpha \cdot \bar{x} \in S$ for all $\bar{x} \in S$ and scalar α.

(ii) $\bar{x} + \bar{y} \in S$ for all $\bar{x}, \bar{y} \in S$.

Remark. (i), (ii) implies S must contain the zero vector $\bar{0}$ in V

e.g. $V = \mathbb{R}^2$ (Euclidean Vector Space in the usual sense)

$\bar{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$: vector in $\mathbb{R}^2$.

Let $S_1 = \left\{ \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \middle| \ x_1 = x_2 \right\} \subset \mathbb{R}^2$
(subset notation)

1. Non-empty: $\bar{0} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \in S_1$ since $0 = 0$.

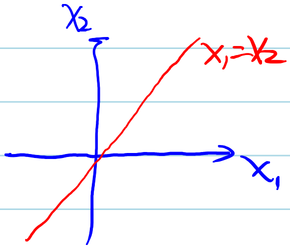
2. If $\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \in S_1$ then $\alpha \cdot \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} \alpha x_1 \\ \alpha x_2 \end{pmatrix}$ also satisfies $\alpha x_1 = \alpha x_2$

Therefore, $\alpha \cdot \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \in S_1$

3. $\bar{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \bar{y} = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \in S_1$, $\begin{matrix} x_1 = x_2 \\ y_1 = y_2 \end{matrix}$ $\Rightarrow \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} x_1 + y_1 \\ x_2 + y_2 \end{pmatrix}$, $x_1 + y_1 = x_2 + y_2$

$\Rightarrow \bar{x} + \bar{y} \in S_1$

Remark. S_1 represents the straightline passing through the origin
One can check that any straightline passing through the origin
is a subspace of $\mathbb{R}^2$.



e.g. $S_2 = \left\{ \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \middle| \ x_2 = 1 \right\}$ (or write $S_2 = \left\{ \begin{pmatrix} x_1 \\ 1 \end{pmatrix} \right\}$)

is NOT a subspace.

$\begin{pmatrix} 0 \\ 1 \end{pmatrix} \in S_2$, while $2 \cdot \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 2 \end{pmatrix} \notin S_2$ (Not closed under scalar multiplication)

or $\begin{pmatrix} 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \end{pmatrix} \in S_2$, while $\begin{pmatrix} 0 \\ 1 \end{pmatrix} + \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \end{pmatrix} \notin S_2$ (Not closed under addition) eg. $V = P_n$: (Polynomial vector space of degree less than n)

$S = \{$ all ploynomials $p(x)$ of degree less than n and $p(0)=0 \}$.

$= \{ a_0 + a_1 x + \cdots + a_{n-1} \cdot x^{n-1} \mid a_0 = 0 \}$

S is nonempty since zero polynomial $\in S$.

$p(x), q(x) \in S \Rightarrow p(0)=0, \; q(0)=0.$

$\alpha \cdot p(x) \in S$ since $\alpha \cdot p(0) = 0$ for any α.

$p(x) + q(x) \in S$ since $p(0) + q(0) = 0$

eg. $V = \mathbb{R}^n$

Let A be a $m \times n$ matrix. Let

$N(A) = \{ \bar{x} \in \mathbb{R}^n \mid A\bar{x} = \bar{0} \}$ be the collection of all solutions to the homogeneous system. $N(A)$ is called the null space of the matrix A.

Claim: $N(A)$ is a subspace of $\mathbb{R}^n$.

1. Nonempty: $\bar{0} \in N(A)$ since $A\bar{0} = \bar{0}$.

2. $\bar{x} \in N(A) \Rightarrow A\bar{x} = \bar{0} \Rightarrow A(\alpha\bar{x}) = \alpha(A\bar{x}) = \alpha\bar{0} = \bar{0}$
$\Rightarrow \alpha\bar{x} \in N(A)$.

3. $\bar{x}, \bar{y} \in N(A) \Rightarrow A\bar{x} = \bar{0}, A\bar{y} = \bar{0} \Rightarrow A(\bar{x} + \bar{y}) = A\bar{x} + A\bar{y}$
$= \bar{0} + \bar{0} = \bar{0}$

$\Rightarrow \bar{x} + \bar{y} \in N(A)$

Therefore, $N(A)$ is a nonempty subset of $\mathbb{R}^n$, which is closed under the addition and (scalar) multiplication of $\mathbb{R}^n$.

$\boxed{\text{eg}}$ Let $A=\begin{bmatrix}1&1&1&0\\2&1&0&1\end{bmatrix}$. Determine $N(A)$. (Find $N(A)$).

Solution: Solve $A\bar{x}=\bar{0}$.

$$\left[\begin{array}{cccc|c}1&1&1&0&0\\2&1&0&1&0\end{array}\right]\xrightarrow{\text{Row②}-2\text{Row①}}\left[\begin{array}{cccc|c}1&1&1&0&0\\0&-1&-2&1&0\end{array}\right]$$

$$\xrightarrow{(-1)\cdot\text{Row②}}\left[\begin{array}{cccc|c}1&1&1&0&0\\0&1&2&-1&0\end{array}\right]$$

$$\xrightarrow{\text{Row①}-\text{Row②}}\left[\begin{array}{cccc|c}1&0&-1&1&0\\0&1&2&-1&0\end{array}\right]$$

(lead variables: x_1, x_2. Free variables: x_3, x_4)

$$\begin{cases}x_1-x_3+x_4=0\\x_2+2x_3-x_4=0\end{cases}\Rightarrow\begin{cases}x_1=x_3-x_4\\x_2=-2x_3+x_4.\end{cases}$$

Set $x_3=\alpha$, $x_4=\beta$.

Then $$\bar{x}=\begin{pmatrix}x_1\\x_2\\x_3\\x_4\end{pmatrix}=\begin{pmatrix}\alpha-\beta\\-2\alpha+\beta\\\alpha\\\beta\end{pmatrix}=\begin{pmatrix}\alpha\\-2\alpha\\\alpha\\0\end{pmatrix}+\begin{pmatrix}-\beta\\\beta\\0\\\beta\end{pmatrix}=\alpha\cdot\begin{pmatrix}1\\-2\\1\\0\end{pmatrix}+\beta\cdot\begin{pmatrix}-1\\1\\0\\1\end{pmatrix}$$

Then

$$N(A)=\left\{\alpha\begin{pmatrix}1\\-2\\1\\0\end{pmatrix}+\beta\begin{pmatrix}-1\\1\\0\\1\end{pmatrix}\ \middle|\ \alpha,\beta\in\mathbb{R}.\right\},$$

meaning that $N(A)$ consists of all vectors of the form $\alpha\begin{pmatrix}1\\-2\\1\\0\end{pmatrix}+\beta\cdot\begin{pmatrix}-1\\1\\0\\1\end{pmatrix}$ ▨.

Recall the definition in Sec 1.4. The above sum is called a linear combination of the vectors $\begin{pmatrix}1\\-2\\1\\0\end{pmatrix}$ and $\begin{pmatrix}-1\\1\\0\\1\end{pmatrix}$.

Def. Let $v_1, v_2, \cdots, v_n$ be vectors in a vector space V.
The span of $v_1, \cdots, v_n$ is defined to be the (sub)set of V consisting of all linear combinations of $v_1, \cdots, v_n$.
We denote the span of $v_1, \cdots, v_n$ by

$$\mathrm{Span}(v_1, \cdots, v_n) = \{ \alpha_1 \cdot v_1 + \alpha_2 v_2 + \cdots + \alpha_n \cdot v_n \mid \alpha_1, \cdots, \alpha_n \in \mathbb{R} \}.$$

Remark. In the previous example, we see that $N(A) = \mathrm{Span}\left(\begin{pmatrix} 1 \\ -2 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 1 \\ 0 \\ 1 \end{pmatrix} \right)$.

Theorem 3.2.1. $\mathrm{Span}(v_1, v_2, \cdots, v_n)$ is a subspace of V for any $v_1, \cdots, v_n \in V$.

Def. If $\mathrm{Span}(v_1, v_2, \cdots, v_n) = V$, then we say $\{v_1, \cdots, v_n\}$ is a spanning set V. $\mathrm{Span}(v_1, \cdots, v_n) = V$ iff any vector in V can be written as a linear combination of $v_1, \cdots, v_n$. We also say the vectors $v_1, \cdots, v_n$ span V.

e.g. (trivial span)
$\bar{e}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \bar{e}_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \bar{e}_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$ span $\mathbb{R}^3$ since for any $\bar{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \in \mathbb{R}^3$, $\bar{x} = x_1 \bar{e}_1 + x_2 \cdot \bar{e}_2 + x_3 \cdot \bar{e}_3$

e.g. Is $\{ (1,1,1)^T, (1,1,0)^T, (1,0,0)^T \}$ a spanning set of $\mathbb{R}^3$?

The problem is equivalent to ask: For any $\bar{b} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} \in \mathbb{R}^3$, are there x_1, x_2, x_3 such that $\bar{b} = x_1 \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + x_2 \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + x_3 \cdot \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$

Notice that in the form of the matrix equation:

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}.$$

For any $\bar{b} \in \mathbb{R}^3$, the system always has a (unique) solution since the coefficient matrix $A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$ is nonsingular (by checking $\det A = -1 \neq 0$).

Therefore, the three vectors span $\mathbb{R}^3$.