Chapter 3. Vector space.

Sec 3.1. Definition and Examples.

Def. Vector Space = Set V with addition $+$ and scalar multiplication $\cdot$

The elements in the set are called "vectors", denoted by $\bar{x}$, $\bar{y}$, etc......

The two operations satisfy the following axioms:

For vectors $\bar{x}, \bar{y}, \bar{z} \in V$, and scalars $\alpha, \beta \in \mathbb{R}$,

C1: If $\bar{x} \in V$, then $\alpha \cdot \bar{x} \in V$.

C2: If $\bar{x}, \bar{y} \in V$, then $\bar{x} + \bar{y} \in V$.

A1: $\bar{x} + \bar{y} = \bar{y} + \bar{x}$

A2: $(\bar{x} + \bar{y}) + \bar{z} = \bar{x} + (\bar{y} + \bar{z})$

A3: There is a zero vector $\bar{0} \in V$ such that $\bar{x} + \bar{0} = \bar{x}$

A4: For each $\bar{x}$, there is a $-\bar{x}$ such that $\bar{x} + (-\bar{x}) = \bar{0}$

A5: $\alpha(\bar{x} + \bar{y}) = \alpha \cdot \bar{x} + \alpha \cdot \bar{y}$

A6: $(\alpha + \beta)\bar{x} = \alpha \bar{x} + \beta \cdot \bar{y}$

A7: $(\alpha\beta)\bar{x} = \alpha(\beta \bar{x})$

A8: $1 \cdot \bar{x} = \bar{x}$

T1: $0 \cdot \bar{x} = \bar{0}$

T2: $\bar{x} + \bar{y} = \bar{0} \Rightarrow \bar{y} = -\bar{x}$

T3: $(-1) \cdot \bar{x} = -\bar{x}$

e.g. Euclidean Vector Space $\mathbb{R}^n$.

$V = \mathbb{R}^2$ (the collection of all vectors $\bar{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$).

equipped with

(vector) addition: $\bar{x} + \bar{y} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} x_1+y_1 \\ x_2+y_2 \end{pmatrix}$.

and

scalar multiplication: $\alpha \bar{x} = \alpha \cdot \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} \alpha x_1 \\ \alpha x_2 \end{pmatrix}$.

(zero vector) $\bar{0} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$. (additive inverse): $-\bar{x} = \begin{pmatrix} -x_1 \\ -x_2 \end{pmatrix}$.

e.g. $m \times n$ matrix space: $M(m,n)$ $\quad (\mathbb{R}^m \times \mathbb{R}^n, \ \mathbb{R}^{m \times n})$

$V = M(3,2)$: the collection of all 3×2 matrices $A = (a_{ij})$

equipped with.

(matrix) addition: $A + B = (a_{ij}) + (b_{ij}) = (a_{ij} + b_{ij})$

scalar multiplication: $\alpha A = \alpha (a_{ij}) = (\alpha \cdot a_{ij})$.

zero vector (zero matrix): $O = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$

(additive inverse): $-A = (-a_{ij})$

Remark: matrix multiplication $A \cdot B$ is not needed for a vector space.

e.g. Continuous function space over interval $[a,b]$: $C[a,b]$.

$V = C[0,1]$: all continuous functions $f(x)$ on $[0,1]$

addition: $(f+g)(x) = f(x) + g(x)$

scalar multiplication: $(\alpha f)(x) = \alpha \cdot f(x)$.

zero function: $f=0$ ($f(x)=0$ for all $x \in [0,1]$).

e.g. Polynomials of degree less than n: P_n

$V = P_3$: all polynomials $p(x)$ of degree less than 3.

e.g. $p(x) = 3+x$, $q(x) = -2-x+x^2$

$p(x) = a_0 + a_1 x + a_2 x^2$ (general form).

addition: $(p+q)(x) = p(x) + q(x)$

scalar multiplication: $(\alpha p)(x) = \alpha \cdot p(x)$

zero polynomial: $p(x) = 0$ (for all x).

Remark: Some textbook denotes the above example by P_2. And in general denote P_n the polynomials of degree less or equal to n.

Remark: The space with all polynomials of degree EXACTLY n is NOT a vector space. for many reasons. e.g. It is not closed under addition and has no zero vector.

$p(x) = x - x^2$, $q(x) = x^2$, $p(x) + q(x) = x$ has degree one (not two)