Sec 2.2. Properties of Determinants

$\boxed{\text{Thm 2.2.2.}}$ A is singular if and only if $\det A = 0$

$\boxed{\text{Thm 2.2.3}}$ If A and B are $n\times n$ matrices, then $\det(AB) = \det A \cdot \det B$.

Remark. In general, $AB \neq BA$. However, $\det(AB) = \det(BA)$ for any A, B.

The proofs of the above two theorems are based on the following properties of elementary matrices.

Proposition.

$$\det(EA) = \begin{cases} -\det A & \text{if } E \text{ is of type ①} \\ \alpha \cdot \det A & \text{if } E \text{ is of type ②} \\ \det A & \text{if } E \text{ is of type ③}. \end{cases} \quad \text{elementary matrix.}$$

$$= \det E \, \det A$$

$$= \det(AE) \quad (\text{see proof in the end})$$

Type ①:

$$\underbrace{\det\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{E} = \det\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \cdot 1 = -1.$$

(Expanding w.r.t. the last row)

E type ①, $A = (a_{ij})$ 3×3.

$$\det(EA) = \det\left(\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}\right)$$

$$= \det\begin{bmatrix} a_{21} & a_{22} & a_{23} \\ a_{11} & a_{12} & a_{13} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

Expanding with respect to the SECOND row

$$= (-1)^{2+1} \cdot a_{11} \cdot \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} + (-1)^{2+2} \cdot a_{12} \cdot \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + (-1)^{2+3} a_{13} \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

$$= -a_{11} \cdot \det M_{11} + a_{12} \cdot \det M_{12} - a_{13} \cdot \det M_{13}$$

where $\det M_{1i}$ is the minor of the original A at a_{1i}

$$= -a_{11} \cdot A_{11} - a_{12} A_{12} - a_{13} A_{13}$$

A_{1i} is the cofactor

$$= -\det A.$$

Type ②: $E = \begin{pmatrix} 1 & & & & 0 \\ & \ddots & & & \\ & & \alpha & & \\ & 0 & & \ddots & \\ & & & & 1 \end{pmatrix}$ diagonal. type ② $\alpha \neq 0$.

$\det A = 1 \cdots 1 \cdot \alpha \cdots\cdots 1 = \alpha$.

e.g. $E = \begin{bmatrix} -2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$

$$\det(EA) = \det \begin{bmatrix} -2a_{11} & -2a_{12} & -2a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = -2a_{11} A_{11} - 2a_{12} A_{12} - 2a_{13} A_{13}$$
$$= (-2) \cdot \det A.$$

Caution: For a $n \times n$ matrix A and a scalar $\alpha \neq 0$,
$\det(\alpha A) = \alpha^n . \det A$.

e.g. $A = (a_{ij})$ 3×3.
$$\det((-2)A) = \det \cdot \begin{bmatrix} -2a_{11} & -2a_{12} & -2a_{13} \\ -2a_{21} & -2a_{22} & -2a_{23} \\ -2a_{31} & -2a_{32} & -2a_{33} \end{bmatrix} = (-2)^3 \cdot \det A.$$

Type ③: $E = \begin{bmatrix} 1 & \ddots & \alpha \\ 0 & & \ddots \\ & & 1 \end{bmatrix}$ or $E = \begin{bmatrix} 1 & & 0 \\ & \ddots & \\ \alpha & \cdots & 1 \end{bmatrix}$ is upper/lower triangular.

$\det E = 1$. The proof for $\det(EA) = \det A$ requires an additional lemma.

e.g. $A = \begin{bmatrix} 2 & 1 & 3 \\ 4 & 2 & 1 \\ 6 & -3 & 4 \end{bmatrix} \longrightarrow \begin{bmatrix} 2 & 1 & 3 \\ 0 & 0 & -5 \\ 0 & -6 & -5 \end{bmatrix}$ type ③. operations on row 2 and row 3.

$$\det A = \begin{vmatrix} 2 & 1 & 3 \\ 4 & 2 & 1 \\ 6 & -3 & 4 \end{vmatrix} = \begin{vmatrix} 2 & 1 & 3 \\ 0 & 0 & -5 \\ 0 & -6 & -5 \end{vmatrix} = 2 \cdot \begin{vmatrix} 0 & -5 \\ -6 & -5 \end{vmatrix}$$ Expanding by the first column

$$= 2 \cdot (0 - (-5)(-6)) = -60.$$

Proof of Thm 2.2.2.

Recall in Sec 1.5, we showed that if A is invertible, then A is row equivalent to I. More precisely,

$$A = E_k \cdots E_2 \cdot E_1, \text{ where } E_1, \dots, E_k \text{ are all elementary matrices.}$$

Therefore,

$$\det(A) = \det(E_k) \cdot \det(E_{k-1} \cdots E_1)$$
$$= \det(E_k) \det(E_{k-1}) \cdot \det(E_{k-2} \cdots E_1)$$
$$\cdots = \det(E_k) \cdot \det(E_{k-1}) \cdots \det(E_2) \cdot \det E_1 \neq 0,$$

since the determinants of all the elementary matrices are non-zero.

Proof of Thm 2.2.3.

case 1: If B is singular, then AB is singular. (H/W 18* in Sec 1.5).
Therefore, $\det(AB) = 0 = \det A \cdot \det B$.

case 2: If B is non-singular, then $B = E_k \cdots E_1$, where $E_1, \dots, E_k$ are elementary matrices.

$$AB = A \cdot E_k \cdot E_{k-1} \cdots E_1.$$

$$\Rightarrow \det(AB) = \det(A E_k \cdots E_2) \cdot \det(E_1)$$
$$= \det(A E_k \cdots E_3) \cdot \det E_2 \cdot \det E_1$$
$$\cdots = \det A \cdot (\det E_k) \cdots (\det E_2)(\det E_1)$$
$$= \det A \cdot \det B.$$

Remark. Proof of $\det(AE) = \det E \cdot \det A$.

$$\det(AE) = \det((AE)^T) = \det(E^T A^T)$$
$$= \det(E^T) \cdot \det(A^T)$$
$$= \det E \cdot \det A$$

e.g. (Ex 3, (d))

$$\begin{vmatrix} 2 & 1 & 1 \\ 4 & 3 & 5 \\ 2 & 1 & 2 \end{vmatrix} = \begin{vmatrix} 2 & 1 & 1 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{vmatrix}$$

Row operation type ③:
Row 2 − 2·Row 1.
Row 3 − Row 1.

$$= 2 \cdot 1 \cdot 1 = 2.$$

Sec 2.3*. (Not required).

Let $A=(a_{ij})$ be a $n\times n$ matrix, and A_{ij} be its cofactor (see def in Sec 2.1)

Def. Adjoint of A:

$$\operatorname{adj} A = (A_{ji}) = \begin{bmatrix} A_{11} & A_{21} & \cdots & A_{n1} \\ A_{12} & A_{22} & \cdots & A_{n2} \\ A_{13} & \vdots & & \\ \vdots & \vdots & & \\ A_{1n} & A_{2n} & \cdots & A_{nn} \end{bmatrix}$$

Formula of the inverse by the adjoint for $n\times n$ matrix.

(1): $$A^{-1} = \frac{1}{\det A}\cdot \operatorname{adj} A \qquad (\text{when } \det A \neq 0).$$

Remark. 2×2 case. $A=\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22}\end{pmatrix}$, $\quad A_{11}=a_{22} \quad A_{12}=-a_{21}$, $\quad A_{21}=-a_{12} \quad A_{22}=a_{11}$

$$\operatorname{adj} A = \begin{pmatrix} a_{22} & -a_{21} \\ -a_{12} & a_{11}\end{pmatrix}.$$

$$A^{-1} = \frac{1}{\det A}\cdot\begin{pmatrix} a_{22} & -a_{21} \\ -a_{12} & a_{11}\end{pmatrix}$$ is a special case of formula (1).

Thm 2.3.1. Cramer's rule. $A\bar{x}=\bar{b}$, $\bar{x}=\begin{pmatrix}x_1\\ \vdots \\ x_n\end{pmatrix}$, $\bar{b}=\begin{pmatrix}b_1\\ \vdots \\ b_n\end{pmatrix}$

If A is nonsingular, then

$$x_i = \frac{\det\begin{bmatrix} a_1 & \cdots & b_1 & \cdots & a_{1n} \\ \vdots & & \vdots & & \\ a_{n1} & \cdots & b_n & \cdots & a_{nn}\end{bmatrix}}{\det A}.$$

the matrix is obtained by replacing the ith column of A by $\bar{b}$.

Proof. $\bar{x}=A^{-1}\bar{b} = \frac{1}{\det A}(\operatorname{adj} A\cdot \bar{b})$

$$\begin{bmatrix} A_{11} & A_{12} & \cdots & A_{1n} \\ \vdots & & & \vdots \\ A_{1i} & A_{2i} & \cdots & A_{ni} \\ & & & \vdots \\ A_{1n} & \cdots & & A_{nn}\end{bmatrix}\cdot\begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n\end{bmatrix}$$

The ith row of the product $\operatorname{adj} A\cdot\bar{b}$ is:

$$b_1\cdot A_{1i} + b_2\cdot A_{2i} + \cdots + b_n\cdot A_{ni}$$

which is the determinant of the following matrix (by expanding w.r.t. the i-th column)

$$\begin{bmatrix} a_{11} & \cdots & b_1 & \cdots & a_{1n} \\ a_{21} & \vdots & b_2 & & a_{2n} \\ a_{31} & & \vdots & & \vdots \\ \vdots & \vdots & \vdots & & \vdots \\ a_{n1} & & b_n & \cdots & a_{nn}\end{bmatrix}$$

where b_j is at (j,i) position of $A=(a_{ij})$ and the cofactor is A_{ji}