Sec 1.5 Elementary Matrices

3 types of $n\times n$ matrices

Type ①:

$$E=\begin{array}{cl}\begin{pmatrix}1 & & & & & & \\ & 1 & & & & & \\ & & \ddots & & & & \\ & & & 0 & & 1 & \\ & & & & \ddots & & \\ & & & 1 & & 0 & \\ & & & & & & \ddots \\ & & & & & & & 1\end{pmatrix} & \begin{array}{l} \\ \\ \\ \text{-- } i\text{-raw} \\ \\ \text{-- } j\text{-raw} \\ \\ \\ \end{array}\end{array}$$

Type ②

$$E=\begin{array}{cl}\begin{pmatrix}1 & & & & & \\ & 1 & & & 0 & \\ & & \ddots & & & \\ & & & \alpha & & \\ & 0 & & & \ddots & \\ & & & & & 1\end{pmatrix} & \begin{array}{l} \\ \\ \\ \text{-- } i\text{-raw} \\ \\ \\ \end{array}\end{array}, \quad \alpha\neq 0$$

Type ③

$$E=\begin{array}{cl}\begin{pmatrix}1 & & & & 0 \\ & 1 & \cdots & \alpha & \\ & & \ddots & \vdots & \\ & 0 & & 1 & \\ & & & & \ddots \\ & & & & & 1\end{pmatrix} & \begin{array}{l} \\ \text{-- } (i,j) \text{ entry } \alpha\neq 0. \\ \\ \text{-- } j\text{-raw.} \\ \\ \\ \end{array}\end{array}$$

Given $n \times n$ matrices A

EA represents the matrix obtained by applying the elementary row operation to A, where the row operation has the same type of E.

Remark: AE will be the version for column operation. We will mainly focus on row operation and left product.

Properties of the three types of E.

Type ①: EA, interchange row-i and row-j.

$$E^{-1} = E = \begin{array}{c} \begin{pmatrix} 1 & & & & & & \\ & \ddots & & & & & \\ & & 0 & & 1 & & \\ & & & \ddots & & & \\ & & 1 & & 0 & & \\ & & & & & \ddots & \\ & & & & & & 1 \end{pmatrix} \begin{array}{l} \\ \\ - i \\ \\ - j \\ \\ \\ \end{array} \end{array}$$

Type ②: EA.

multiplying raw-i by α

$$E = \begin{array}{c} \begin{pmatrix} 1 & & & & \\ & \ddots & & & \\ & & \alpha & & \\ & & & \ddots & \\ & & & & 1 \end{pmatrix} \end{array} \begin{array}{c} \\ \\ i \\ \\ \\ \end{array}, \quad E^{-1} = \begin{array}{c} \begin{pmatrix} 1 & & & & \\ & \ddots & & & \\ & & \frac{1}{\alpha} & & \\ & & & \ddots & \\ & & & & 1 \end{pmatrix} \end{array} \begin{array}{c} \\ \\ i \\ \\ \\ \end{array}$$

Type ③: EA.

adding $\alpha \times$ raw-j to raw-i

(replacing raw-i by raw-i + $\alpha \times$ raw-j).

$$E = \begin{pmatrix} 1 & & & & & \\ & \ddots & & & & \\ & & 1 & & \alpha & \\ & & & \ddots & & \\ & & & & 1 & \\ & & & & & \ddots & \\ & & & & & & 1 \end{pmatrix} \begin{array}{c} \\ \\ i \\ \\ j \\ \\ \\ \end{array}, \quad E^{-1} = \begin{pmatrix} 1 & & & & & \\ & \ddots & & & & \\ & & 1 & & -\alpha & \\ & & & \ddots & & \\ & & & & 1 & \\ & & & & & \ddots & \\ & & & & & & 1 \end{pmatrix}$$

Remark:

For $E = \begin{pmatrix} 1 & & & & \\ & \ddots & & & \\ & & 1 & & \\ & & \alpha & \ddots & \\ & & & & 1 \end{pmatrix} \begin{array}{c} \\ \\ i \\ j \\ \\ \end{array}$, the operation is adding $\alpha \times$ raw-i to raw-j.

Def. (Row equivalent).

B is said to be row equivalent to A if there are elementary matrices $E_1, E_2, \cdots, E_k$ such that

$$B = E_k \cdot E_{k-1} \cdots E_1 \cdot A.$$

$$(\text{or } E_k \cdot E_{k-1} \cdots E_1 \cdot B = A)$$

Thm (1.5.2)

The following are equivalent:

(a) A is nonsingular

(b) $A\bar{x} = \bar{0}$ has no nontrivial solution

(c). A is row equivalent to I.

Remark: $A \cdot \bar{x} = \bar{0}$ always has a solution $\bar{x} = \bar{0} = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix}$ which we call trivial solution.

e.g. Using row operation to find the inverse of a given matrix.

$$A=\begin{bmatrix}2 & 5\\ 1 & 3\end{bmatrix}.$$

$$\left[\begin{array}{cc:cc}2 & 5 & 1 & 0\\ 1 & 3 & 0 & 1\end{array}\right] \xrightarrow{E_1=\begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix}} \left[\begin{array}{cc:cc}1 & 3 & 0 & 1\\ 2 & 5 & 1 & 0\end{array}\right]$$

$$\xrightarrow{E_2=\begin{bmatrix}1 & 0\\ -2 & 1\end{bmatrix}} \left[\begin{array}{cc:cc}1 & 3 & 0 & 1\\ 0 & -1 & 1 & -2\end{array}\right] \xrightarrow{E_3=\begin{bmatrix}1 & 0\\ 0 & -1\end{bmatrix}}$$

$$\left[\begin{array}{cc:cc}1 & 3 & 0 & 1\\ 0 & 1 & -1 & 2\end{array}\right] \xrightarrow{E_4=\begin{bmatrix}1 & -3\\ 0 & 1\end{bmatrix}} \left[\begin{array}{cc:cc}1 & 0 & 3 & -5\\ 0 & 1 & -1 & 2\end{array}\right]$$

$$\underbrace{\begin{matrix}3 & -5\\ -1 & 2\end{matrix}}_{A^{-1}.}$$

$$A^{-1}=\begin{bmatrix}3 & -5\\ -1 & 2\end{bmatrix}=E_4\cdot E_3\cdot E_2\cdot E_1 \text{ since}$$

$$(E_4\cdot E_3\cdot E_2\cdot E_1)\cdot A=I, \text{ and } A=(E_4\cdot E_3\cdot E_2\cdot E_1)^{-1}=E_1^{-1}E_2^{-1}E_3^{-1}E_4^{-1}$$