Sec 1.4 Matrix Algebra

Thm 1.4.1. (Algebraic Rules)

For scalars α, β, and matrices A, B, C, we have.

1. $A+B=B+A$; 2. $(A+B)+C=A+(B+C)$
3. $(AB)C=A(BC)$; 4. $A(B+C)=AB+AC$
5. $(A+B)C=AC+BC$; 6. $(\alpha\beta)A=\alpha(\beta A)$
7. $\alpha(AB)=(\alpha A)B=A(\alpha B)$ 8. $(\alpha+\beta)A=\alpha A+\beta A$
9. $\alpha(A+B)=\alpha A+\alpha B$.

Warning: Most rules above are exactly the same as those for scalars. But be cautious about the order of the matrices product.
In particular, $\boxed{AB \neq BA}$. in general.

e.g. $A=\begin{bmatrix}1 & 1\\ 0 & 1\end{bmatrix}$ $B=\begin{bmatrix}1 & 2\\ 3 & 4\end{bmatrix}$

$$AB=\begin{bmatrix}1 & 1\\ 0 & 1\end{bmatrix}\begin{bmatrix}1 & 2\\ 3 & 4\end{bmatrix}=\begin{bmatrix}4 & 6\\ 3 & 4\end{bmatrix} \neq BA=\begin{bmatrix}1 & 2\\ 3 & 4\end{bmatrix}\begin{bmatrix}1 & 1\\ 0 & 1\end{bmatrix}=\begin{bmatrix}1 & 3\\ 3 & 7\end{bmatrix}$$

Notation 1. Matrix power A^k, where k is a positive integer and A is a $\boxed{n\times n \text{ square matrix}}$

$$A^k=\underbrace{A\cdot A\cdot A\cdots\cdots A}_{k \text{ times}} \quad (\text{product of } k\text{-copies of } A)$$

e.g. $A=\begin{bmatrix}1 & 1\\ 1 & 1\end{bmatrix}$. Then $A^2=A\cdot A=\begin{bmatrix}1 & 1\\ 1 & 1\end{bmatrix}\begin{bmatrix}1 & 1\\ 1 & 1\end{bmatrix}=\begin{bmatrix}2 & 2\\ 2 & 2\end{bmatrix}$.

A^k can be computed inductively from A^{k-1}, since
$A^k=A^{k-1}\cdot A$.

e.g. $A=\begin{bmatrix}1 & 1\\ 1 & 1\end{bmatrix}$. Then $A^3=A^2\cdot A=\begin{bmatrix}2 & 2\\ 2 & 2\end{bmatrix}\begin{bmatrix}1 & 1\\ 1 & 1\end{bmatrix}=\begin{bmatrix}4 & 4\\ 4 & 4\end{bmatrix}$

Optional Problem: For $A=\begin{bmatrix}1 & 1\\ 1 & 1\end{bmatrix}$, find the general expression of A^n for all $n\geq 2$.
(prove)

Notation 2 The $n\times n$ identity matrix I is defined to be $I=(\delta_{ij})$, where

$$\delta_{ij}=\begin{cases}1 & \text{if } i=j\\ 0 & \text{if } i\neq j.\end{cases}$$

e.g. 2×2 identity matrix $I = \begin{pmatrix} \delta_{11} & \delta_{12} \\ \delta_{21} & \delta_{22} \end{pmatrix}$. By the definition of δ_{ij}, we have

$\delta_{11}=1$, $\delta_{22}=1$ and $\delta_{12}=0$, $\delta_{21}=0$

Therefore, $I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$.

In general, the $n\times n$ identity matrix $I = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \ddots & \vdots \\ 0 & 0 & \ddots & \ddots & 0 \\ \vdots & & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & 0 & 1 \end{pmatrix}$

Remark: Some books also write I_n to indicate the size of the identity.

e.g. $I_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$.

Identity matrix I plays the role of scalar 1 in matrix product in the following sense

For any $m\times n$ matrix A, and the $n\times n$ identity I, we have

$$A \cdot I = A.$$

For a $m\times n$ A, and the $m\times m$ identity I, we have

$$I \cdot A = A.$$

e.g. $A = \begin{bmatrix} 1 & 0 & -1 \\ 2 & 3 & 0 \end{bmatrix}$ (2×3 matrix)

2×2 I. $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 & -1 \\ 2 & 3 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & -1 \\ 2 & 3 & 0 \end{bmatrix}$

3×3 I $\begin{bmatrix} 1 & 0 & -1 \\ 2 & 3 & 0 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & -1 \\ 2 & 3 & 0 \end{bmatrix}$

Def. (Inverse of a square matrix)

For a $n\times n$ matrix A, if there is a matrix B such that $AB=BA=I$, then we say A is invertible (or nonsingular). We call B the inverse of A, and denote it by A^{-1}

Remark. The inverse matrix is unique. Either A has no inverse or there is one and only one inverse.

e.g. Let $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$. One can check that $\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$ and

$\begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$. Therefore, $\begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}$ is the inverse of $\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$

Def. If A has no inverse matrix, then A is called singular or not invertible.

e.g. $A=\begin{bmatrix}1 & 1\\ 0 & 0\end{bmatrix}$. For any $B=\begin{bmatrix}b_{11} & b_{12}\\ b_{21} & b_{22}\end{bmatrix}$,

$$A\cdot B=\begin{bmatrix}1 & 1\\ 0 & 0\end{bmatrix}\begin{bmatrix}b_{11} & b_{12}\\ b_{21} & b_{22}\end{bmatrix}=\begin{bmatrix}b_{11} & b_{12}+b_{22}\\ 0 & 0\end{bmatrix}$$ can not be the identity matrix.

Therefore, there is no matrix B such that $AB=BA=I$, ie.,
A has no inverse and we say $\begin{bmatrix}1 & 1\\ 0 & 0\end{bmatrix}$ is singular or not invertible.

• Important formulas for inverse and transpose

Thm 1.4.2 If A and B are both $n\times n$ invertible matrices, then AB is also invertible. Moreover, we have

$$\boxed{(AB)^{-1}=B^{-1}\cdot A^{-1}}$$

Rules for transposes:

1. $(A^T)^T=A$
2. $(\alpha A)^T=\alpha\cdot A^T$
3. $(A+B)^T=A^T+B^T$
4. $(AB)^T=B^T\cdot A^T$

Other important formulas contained in the Homework.

HW12. Let $A=\begin{bmatrix}a_{11} & a_{12}\\ a_{21} & a_{22}\end{bmatrix}$. If $a_{11}\cdot a_{22}-a_{12}\cdot a_{21}\neq 0$, then A is invertible and $A^{-1}=\dfrac{1}{a_{11}a_{22}-a_{12}a_{21}}\cdot\begin{bmatrix}a_{22} & -a_{12}\\ -a_{21} & a_{11}\end{bmatrix}$

e.g. $A=\begin{bmatrix}2 & 1\\ 4 & 3\end{bmatrix}$. $2\cdot 3-1\cdot 4=2\neq 0$. Then $A^{-1}=\dfrac{1}{2}\begin{bmatrix}3 & -1\\ -4 & 2\end{bmatrix}=\begin{bmatrix}\frac{3}{2} & -\frac{1}{2}\\ -2 & 1\end{bmatrix}$.

Remark. This formula only works for 2×2 matrix.

• If A is nonsingular, then ①: $(A^{-1})^{-1}=A$. (HW 15)
② $(A^T)^{-1}=(A^{-1})^T$ (HW 16)