Sec 1.3. Matrix Arithmetic.

Notations:

1. $m\times n$ matrix $A=(a_{ij})=\begin{bmatrix} a_{11} & \cdots & a_{1n} \\ a_{21} & & \vdots \\ \vdots & & \vdots \\ a_{m1} & \cdots & a_{mn}\end{bmatrix}$

   a_{ij} stands for the entry in ith raw and jth column

   We say A has dimension/size $m\times n$.

2. column vector ($m\times 1$ matrix) in $\mathbb{R}^m$

   $\mathbf{x}=\bar{x}=\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_m\end{pmatrix}$, e.g. $\mathbf{a}=\bar{a}=\begin{pmatrix} 2 \\ 3 \\ -1\end{pmatrix}$ in a column vector in $\mathbb{R}^3$

3. raw vector ($1\times n$ matrix) in $\mathbb{R}^n$

   $\vec{\mathbf{x}}=(x_1, x_2, \cdots, x_n)$, e.g. $\vec{\mathbf{b}}=(0,0,3,1)$ is a raw vector in $\mathbb{R}^4$.

4. $m\times n$ matrix A can be rewriten using its raw or column vectors.

   $A=(a_{ij})$ consists of m raw vectors in $\mathbb{R}^n$

   $\vec{\mathbf{a}}_1=(a_{11}, a_{12}, \cdots, a_{1n})$

   $\vec{\mathbf{a}}_2=(a_{21}, a_{22}, \cdots, a_{2n})$

   $\vdots$

   $\vec{\mathbf{a}}_m=(a_{m1}, a_{m2}, \cdots, a_{mn})$

   $A=\begin{pmatrix} \vec{a}_1 \\ \vec{a}_2 \\ \vdots \\ \vec{a}_m\end{pmatrix}$

   $A=(a_{ij})$ also consists of n column vectors in $\mathbb{R}^m$

   $\bar{a}_1=\begin{pmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1}\end{pmatrix}, \cdots, \bar{a}_j=\begin{pmatrix} a_{1j} \\ a_{2j} \\ \vdots \\ a_{mj}\end{pmatrix} \cdots, \bar{a}_n=\begin{pmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{mn}\end{pmatrix}$ $\quad A=(\bar{a}_1, \bar{a}_2, \cdots, \bar{a}_n)$

e.g. $A=\begin{pmatrix} 2 & 0 & 3 \\ 1 & -1 & 0 \end{pmatrix}$ is a 2×3 matrix

We can write $\vec{a_1}=(2,0,3)$, $\vec{a_2}=(1,-1,0)$, and $A=\begin{pmatrix} \vec{a_1} \\ \vec{a_2} \end{pmatrix}$

or $\bar{a_1}=\begin{pmatrix} 2 \\ 1 \end{pmatrix}$, $\bar{a_2}=\begin{pmatrix} 0 \\ -1 \end{pmatrix}$, $\bar{a_3}=\begin{pmatrix} 3 \\ 0 \end{pmatrix}$, and $A=(\bar{a_1}, \bar{a_2}, \bar{a_3})$

Remark 1. In the textbook, the column vector is in bold face $\mathbf{x}$, here we may write $\bar{x}$ with a bar. The raw vector is with a right arrow $\vec{x}$. The notations are subject to change.

Remark 2. It's fine to use either parentheses or square brackets for matrices and vectors. Some people prefer square ones for matrices and round ones for vectors.

Remark 3. In some textbooks, people also use the following notations to indicate the dimension (size) of a $m\times n$ matrix A.

$$A=(a_{ij})=(a_{ij})_{m\times n}=\{a_{ij}\}=\{a_{ij}\}_{m\times n}$$

We want to define the following arithmetic of matrices.

1. equal: Two $m\times n$ matrices $A=(a_{ij})$ and $B=(b_{ij})$ are called "equal" if $a_{ij}=b_{ij}$ for all i,j.
   We write $A=B$ if they are equal to each other and $A\neq B$ if not. If the sizes of A and B are different, they can not equal.

e.g. $\begin{bmatrix} 2 & 3 & 0 \\ 0 & 0 & 0 \end{bmatrix} \neq \begin{bmatrix} 2 & 3 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, $\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \neq \begin{bmatrix} 0 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$

2 scalar multiplication

For a scalar α and a matrix $A=(a_{ij})$, we define

$$\alpha A := (\alpha \cdot a_{ij}).$$

e.g. $$2\cdot\begin{bmatrix} 2 & 3 \\ 4 & -1 \end{bmatrix} = \begin{bmatrix} 2\cdot 2 & 2\cdot 3 \\ 2\cdot 4 & 2\cdot(-1) \end{bmatrix} = \begin{bmatrix} 4 & 6 \\ 8 & -2 \end{bmatrix}$$

e.g. $$0\cdot\begin{bmatrix} 2 & 3 & 0 \\ -1 & 0 & 5 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

3 Summation (Matrix addition).

Two matrices can be summed up ONLY IF they have the same sizes. For two $m\times n$ matrices

$A=(a_{ij})$ and $B=(b_{ij})$,

$A+B=(a_{ij}+b_{ij})$ is also a $m\times n$ matrix.

e.g. $$\begin{bmatrix} 2 & 0 \\ -1 & 5 \end{bmatrix} + \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} 2+1 & 0+2 \\ -1+3 & 5+4 \end{bmatrix} = \begin{bmatrix} 3 & 2 \\ 2 & 9 \end{bmatrix}$$

e.g. (Sum of two vectors)

$$\begin{bmatrix} 2 & 3 & 5 \end{bmatrix} + \begin{bmatrix} 0 & -1 & 1 \end{bmatrix} = \begin{bmatrix} 2+0 & 3-1 & 5+1 \end{bmatrix}$$
$$= \begin{bmatrix} 2 & 2 & 6 \end{bmatrix}$$

$$\begin{bmatrix} 2 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ -3 \end{bmatrix} = \begin{bmatrix} 2+0 \\ 0-3 \end{bmatrix} = \begin{bmatrix} 2 \\ -3 \end{bmatrix}$$

4. Zero matrix is the matrix whose entries are all zeros.

e.g. $O = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ (2×3 zero matrix)

$O = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ (3×1 zero matrix, or a zero vector)

- Zero matrix plays the role of the scalar 0 in matrix addition in the sense that

$A + O = A = O + A$, for any m×n A and m×n zero matrix O.

★ $\boxed{\text{5. Matrix product.}}$

For a m×n matrix $A = (a_{ij})$ and a n×r matrix $B = (b_{ij})$, the product of A and B, denoted by AB (or $A \cdot B$), is defined to be a m×r matrix $C = (c_{ij})$.

The (i,j)-entry of AB, i.e., c_{ij} is given by:

$$c_{ij} = \sum_{k=1}^{n} a_{ik} \cdot b_{kj}, \quad 1 \le i \le m, \quad 1 \le j \le r.$$

We may write $A \cdot B = \left(\sum_{k=1}^{n} a_{ik} b_{kj} \right)$

Warning: One can compute the product of A and only if the number of $\underbrace{\boxed{\text{columns in A}}}_{n}$ is the same as the number of $\underbrace{\boxed{\text{rows in B}}}_{n}$.

5.1 (Base case. Vector product).

For a raw vector $\vec{a}=(a_1, a_2, \cdots, a_n)$ in R^n

and a column vector $\bar{b}=\begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix}$ also in R^n,

the product $\vec{a}\cdot\bar{b}=(a_1\ a_2 \cdots a_n)\cdot\begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix}=a_1 b_1+a_2 b_2+\cdots+a_n b_n$

This is a special case of the general formula

since $\vec{a}$ is a $1\times n$ matrix and $\bar{b}$ is a $n\times 1$ matrix.

e.g. $(2\quad 3\quad 0)\cdot\begin{pmatrix} -1 \\ 0 \\ 2 \end{pmatrix}=2\cdot(-1)+3\cdot 0+0\cdot 2=-2$.

e.g. (Linear equation with two unknowns)

Let $\vec{a}=(3\quad -5)$ and $\bar{x}=\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$

$\vec{a}\cdot\bar{x}=(3\quad -5)\cdot\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}=3x_1-5x_2$

Then the linear equation $3x_1-5x_2=0$

can be re-written using vector (matrix) product:

$\vec{a}\cdot\bar{x}=0 \iff 3x_1-5x_2=0$

Right hand side can be other numbers: $\vec{a}\cdot\bar{x}=-1 \iff 3x_1-5x_2=-1$.

5.2. Product of a $m\times n$ matrix and a column vector in R^n $(n\times 1)$

$$\begin{bmatrix} 2 & 3 & 0 \\ -1 & 1 & -2 \end{bmatrix}\begin{bmatrix} 2 \\ 0 \\ 3 \end{bmatrix} = \begin{bmatrix} 2\cdot 2+3\cdot 0+0\cdot 3 \\ -1\cdot 2+1\cdot 0+(-2)3 \end{bmatrix} = \begin{bmatrix} 4 \\ -8 \end{bmatrix}$$

size: 2×3 $\quad 3\times 1 \quad 2\times 1$.

For a $m\times n$ matrix $A=(a_{ij})$, write A as the vertical array of its m row vectors

$$A=\begin{pmatrix} \vec{a_1} \\ \vec{a_2} \\ \vdots \\ \vec{a_m} \end{pmatrix}$$ where $\vec{a_i}=(a_{i1}, a_{i2} \cdots a_{in})$ is in $\mathbb{R}^n$.

Let $\bar{x}=\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$ be a column vector in $\mathbb{R}^n$.

Then the matrix product of A and $\overline{x}$ will be a $m \times 1$ column vector:

$$A \cdot \overline{x} = \begin{pmatrix} \vec{a_1} \cdot \overline{x} \\ \vec{a_2} \cdot \overline{x} \\ \vdots \\ \vec{a_m} \cdot \overline{x} \end{pmatrix} = \begin{pmatrix} a_{11} \cdot x_1 + a_{12} x_2 + \cdots + a_{1n} \cdot x_n \\ \vdots \\ a_{m1} \cdot x_1 + a_{m2} x_2 + \cdots + a_{mn} \cdot x_n \end{pmatrix}$$

Notice that the i-th entry of this column vector, $\vec{a_i} \cdot \overline{x}$ is exactly the left hand side of the $m \times n$ linear system, with coeffiient matrix A and unknown $\overline{x}$.

Let $\overline{b} = \begin{pmatrix} b_1 \\ \vdots \\ b_m \end{pmatrix}$ be a column vector in R^m. Then

$$\begin{cases} a_{11} x_1 + \cdots + a_{1n} x_n = b_1 \\ \vdots \\ a_{m1} x_1 + \cdots + a_{mn} x_n = b_m \end{cases} \iff A \cdot \overline{x} = \overline{b}$$

($m \times n$ linear system) (Matrix equation)

5.3. Example of the general case.

e.g. $A = \begin{bmatrix} 2 & 0 \\ -1 & 1 \end{bmatrix}$ $\quad B = \begin{bmatrix} 1 & 0 & 4 \\ 3 & -1 & 5 \end{bmatrix}$.

2×2. $\qquad 2\times 3$.

entry C_{11}: the first row of A and the first column of B

$$A\cdot B = \begin{bmatrix} 2 & 0 \\ -1 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 & 4 \\ 3 & -1 & 5 \end{bmatrix}$$

$2\times 2 \quad 2\times 3$

2×3.

$$= \begin{bmatrix} 2\cdot 1 + 0\cdot 3 & 2\cdot 0 + 0\cdot(-1) & 2\cdot 4 + 0\cdot 5 \\ (-1)\cdot 1 + 1\cdot 3 & (-1)\cdot 0 + 1\cdot(-1) & (-1)\cdot 4 + 1\cdot 5 \end{bmatrix}$$

entry C_{23}: the second row of A and the third column of B.

e.g. $$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 3 & -1 \\ 2 & 0 \\ 5 & 1 \end{bmatrix} = \begin{bmatrix} 3\cdot 1+0+0 & -1\cdot 1+0+0 \\ 0+2\cdot 1+0 & 0+0\cdot 1+0 \\ 0+0+5\cdot 1 & 0+0+1 \end{bmatrix} = \begin{bmatrix} 3 & -1 \\ 2 & 0 \\ 5 & 1 \end{bmatrix}$$

$3\times 3 \qquad 3\times 2$

e.g. $$\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}.$$

Def. If $\bar{a}_1, \bar{a}_2, \cdots, \bar{a}_n$ are n column vectors in $\mathbb{R}^m$ and $c_1, c_2, \cdots, c_n$ are scalars, then the following sum

$$c_1 \cdot \bar{a}_1 + c_2 \cdot \bar{a}_2 + \cdots + c_n \cdot \bar{a}_n$$ is said to be

a $\boxed{\text{linear combination}}$ of vectors $\bar{a}_1, \cdots, \bar{a}_n$.

e.g. Consider a $m \times n$ linear system (in the matrix equation form)

$$A\bar{x} = \bar{b}$$

Notice that $A = (\bar{a}_1 \ \bar{a}_2 \ \cdots \ \bar{a}_n)$, where $\bar{a}_j = \begin{pmatrix} a_{1j} \\ a_{2j} \\ \vdots \\ a_{mj} \end{pmatrix}$ is the jth column vector of A.

Using the definition of matrix product, one can check

$$A\bar{x} = x_1 \bar{a}_1 + x_2 \bar{a}_2 + \cdots + x_n \bar{a}_n,$$ which is a linear combination of $\bar{a}_1, \cdots, \bar{a}_n$. Therefore, the matrix equation $A\bar{x} = \bar{b}$ can be understood as: giving $\bar{a}_1, \cdots, \bar{a}_n$, and $\bar{b}$, find a linear combination of $\bar{a}_1, \cdots, \bar{a}_n$ which equals $\bar{b}$.

e.g. $3 \cdot \begin{pmatrix} 1 \\ 2 \end{pmatrix} + 5 \begin{pmatrix} -2 \\ 1 \end{pmatrix} = \begin{pmatrix} 3 - 5 \cdot 2 \\ 3 \cdot 2 + 5 \cdot 1 \end{pmatrix} = \begin{pmatrix} -7 \\ 11 \end{pmatrix}$ is a linear combination of $\begin{pmatrix} 1 \\ 2 \end{pmatrix}$ and $\begin{pmatrix} -2 \\ 1 \end{pmatrix}$.

$2 \cdot \begin{pmatrix} 1 \\ 2 \end{pmatrix} - \begin{pmatrix} -2 \\ 1 \end{pmatrix} = \begin{pmatrix} 4 \\ 3 \end{pmatrix}$ is also a linear combination of these two vectors.

Def. $\boxed{\text{Transpose}}$ of a $m \times n$ matrix $A = (a_{ij})$ is a $n \times m$ matrix, whose (i,j) entry is a_{ji}. We denote the transpose of A by A^T.

e.g. If $A = \begin{bmatrix} 2 & 3 \\ 0 & 1 \end{bmatrix}$, then $A^T = \begin{bmatrix} 2 & 0 \\ 3 & 1 \end{bmatrix}$

If $B = \begin{bmatrix} 0 & 5 \\ 2 & -1 \\ 1 & 3 \end{bmatrix}$, then $B^T = \begin{bmatrix} 0 & 2 & 1 \\ 5 & -1 & 3 \end{bmatrix}$

For a $n \times n$ matrix A, A^T is also of size $n \times n$.

Def. For a $n \times n$ matrix A, if $A = A^T$, then we say A is symmetric.

e.g. $A = \begin{bmatrix} 2 & 0 & 5 \\ 0 & 3 & 4 \\ 5 & 4 & -1 \end{bmatrix}$ is symmetric; $B = \begin{bmatrix} 2 & 0 \\ 0 & -1 \end{bmatrix}$ is symmetric

$C = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$ is not symmetric since $C^T = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \neq C$.