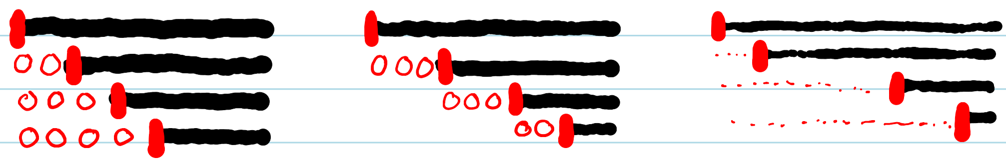
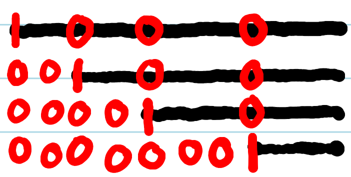
Lec2. Sec 1.2. Row Echelon Form.

Def. Row Echelon Form. (Matrix)



(i) The first nonzero entry in each row is 1. (leading entry)
(ii) Row $k+1$ has more zeros than row k, before leading entry.
(iii) Rows with only zero entries are in the bottom. (not in the middle).

Def. Reduced Row Echelon Form



(i). The matrix is in row echelon form
(ii). The leading entry 1 in each row is the only $\boxed{\text{nonzero entry in the column}}$

e.g.

$$\begin{bmatrix} 1 & 2 & 3 & 0 \\ 0 & 1 & -1 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$ Row Echelon Form.

$$\begin{bmatrix} 1 & 0 & -3 & 0 & 2 \\ 0 & 1 & 2 & 0 & -1 \\ 0 & 0 & 0 & 1 & 5 \end{bmatrix}$$ Reduced Row Echelon Form

The following examples are NOT (Reduced) Row Echelon Form

$$\underbrace{\begin{bmatrix} 2 & 1 \\ 0 & 1 \end{bmatrix} \quad \begin{bmatrix} 1 & 0 & -2 \\ 0 & 0 & 1 \\ 0 & 1 & 3 \end{bmatrix}}_{\text{NOT Row Echelon Form.}} \qquad \underbrace{\begin{bmatrix} 1 & 2 & -2 & 0 & 1 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}}_{\text{Row Echelon Form but NOT Reduced}}$$

- If the augmented matrix of a linear system is in row echelon form, then we can "solve" the system easily.

e.g.

$$\left[\begin{array}{cc|c} 1 & 3 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{array}\right] \Leftrightarrow \begin{cases} x_1 + 3x_2 = +1 \\ \cancel{0 \cdot x_1} + x_2 = -1 \\ \cancel{0 \cdot x_1 + 0 \cdot x_2 = 0} \end{cases}$$

$$\Rightarrow \begin{cases} x_1 - 3 = 1 \\ x_2 = -1 \end{cases} \Rightarrow \begin{cases} x_1 = 4 \\ x_2 = -1 \end{cases}$$ is the only solution

e.g.

$$\left[\begin{array}{ccc|c} 1 & -2 & 4 & 1 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 \end{array}\right] \Leftrightarrow \begin{cases} x_1 - 2x_2 + 4x_3 = 1 \\ 0 \quad 0 \quad x_3 = 3 \\ 0 \quad 0 \quad 0 \quad 0 \end{cases}$$

$$\Rightarrow \begin{cases} x_1 - 2x_2 + 12 = 1 \\ x_3 = 3 \end{cases} \Rightarrow \boxed{\begin{cases} x_1 = 2x_2 - 11 \\ x_3 = 3 \end{cases}}.$$

For any x_2, there will be an associated x_1.

The system has infinitely many solutions, and is consistent.

e.g.

$$\left[\begin{array}{ccc|c} 1 & -2 & 4 & 1 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

No solution. The system is inconsistent.

$\leftarrow$ $0 \cdot x_1 + 0 \cdot x_2 + 0 \cdot x_3 = 1$

similar for

$$\left[\begin{array}{ccc|c} 1 & -2 & 4 & 1 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 5 \\ 0 & 0 & 0 & -2 \end{array}\right] \begin{array}{l} \\ \\ \leftarrow \\ \leftarrow \end{array}$$

No solution. Inconsistent.

e.g.

$$\begin{aligned} x_1 + x_2 + x_3 + x_4 + x_5 &= 2 \\ x_1 + x_2 + x_3 + 2x_4 + 2x_5 &= 3 \\ x_1 + x_2 + x_3 + 2x_4 + 3x_5 &= 2 \end{aligned}$$

$$\left[\begin{array}{ccccc|c} 1 & 1 & 1 & 1 & 1 & 2 \\ 1 & 1 & 1 & 2 & 2 & 3 \\ 1 & 1 & 1 & 2 & 3 & 2 \end{array}\right]$$

$$\longrightarrow \left[\begin{array}{ccccc|c} 1 & 1 & 1 & 1 & 1 & 2 \\ 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 2 & 0 \end{array}\right]$$

$$\longrightarrow \left[\begin{array}{ccccc|c} 1 & 1 & 1 & 1 & 1 & 2 \\ 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & -1 \end{array}\right]$$

row echelon form

$$\longrightarrow \left[\begin{array}{ccccc|c} 1 & 1 & 1 & 1 & 0 & 3 \\ 0 & 0 & 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 0 & 1 & -1 \end{array}\right]$$

$$\longrightarrow \left[\begin{array}{cccccc} 1 & 1 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 0 & 1 & -1 \end{array}\right]$$

R.R.E.F.

$$(*)\begin{cases} x_1 + x_2 + x_3 + 0 + 0 = 1 \\ x_4 = 2 \\ x_5 = -1 \end{cases}$$

$$\Rightarrow \begin{cases} x_1 = -x_2 - x_3 + 1 & \text{(express } x_1 \text{ in terms of } x_2, x_3\text{)} \\ x_4 = 2 \\ x_5 = -1 \end{cases}$$

We call x_1, x_4, x_5 lead variables

We call x_2 and x_3 free variables, and use two Greek letters α, β to represent them.

Then for any $x_2 = \alpha$, $x_3 = \beta$ given,

$x_1 = -\alpha - \beta + 1$

We can express the solution by the 5-tuple

$(-\alpha - \beta + 1, \alpha, \beta, 2, -1)$.

- **Def.** The process to transform a linear system into one whose augmented matrix is in row echelon form is called Gaussian elimination (reduction)
- **Def.** The process of using elementary row operations to transform a matrix into reduced row echelon form is called Gauss-Jordan reduction.

HW 3(a) (b) (c) (d)

4

5 (e) (f)

7*, 8*

Example. Sec 1.2. 5. (d).

$$3x_1 + 2x_2 - x_3 = 4$$

$$x_1 - 2x_2 + 2x_3 = 1$$

$$11x_1 + 2x_2 + x_3 = 14.$$

Reduce it to R.R.F.

$$\left[\begin{array}{ccc|c} 3 & 2 & -1 & 4 \\ 1 & -2 & 2 & 1 \\ 11 & 2 & 1 & 14 \end{array}\right] \xrightarrow{① \leftrightarrow ②} \left[\begin{array}{ccc|c} 1 & -2 & 2 & 1 \\ 3 & 2 & -1 & 4 \\ 11 & 2 & 1 & 14 \end{array}\right]$$

$$\longrightarrow \left[\begin{array}{ccc|c} 1 & -2 & 2 & 1 \\ 0 & 8 & -7 & 1 \\ 0 & 24 & -21 & 3 \end{array}\right]$$

$-11 \quad -22 \quad 22 \quad 11$

$2-(-22) \quad\quad 1-22 \quad 14$

$$\rightarrow \left[\begin{array}{ccc|c} 1 & -2 & 2 & 1 \\ 0 & 8 & -7 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right]$$ 24   -21   3.

$$\rightarrow \left[\begin{array}{ccc|c} 1 & -2 & 2 & 1 \\ 0 & 1 & -\frac{7}{8} & \frac{1}{8} \\ 0 & 0 & 0 & 0 \end{array}\right]$$ consistent.

$$x_1 - 2x_2 + 2x_3 = 1$$

$$x_2 - \frac{7}{8}x_3 = \frac{1}{8}$$

(consistent and has at least one free variable).

Find R.R.E.F.

$$\rightarrow \left[\begin{array}{ccc|c} 1 & 0 & \frac{1}{4} & \frac{3}{4} \\ 0 & 1 & -\frac{7}{8} & \frac{1}{8} \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$x_1 + \frac{1}{4}x_3 = \frac{3}{4}$$
$$x_2 - \frac{7}{8}x_3 = \frac{1}{8}$$
$$\Rightarrow \begin{cases} x_1 = -\frac{1}{4}x_3 + \frac{3}{4} \\ x_2 = -\frac{7}{8}x_3 + \frac{1}{8} \end{cases}$$

x_1, x_2 lead variables, x_3 free v.

all solutions $x_3 \to \alpha$.

$$\left(-\frac{1}{4}\alpha + \frac{3}{4},\ -\frac{7}{8}\alpha + \frac{1}{8},\ \alpha\right).$$

Def. Homogeneous system: if all the right-hand sides b_i are zero.

Inhomogeneous system: if at least one b_i is not zero.

Homogeneous System

$$\begin{cases} x_1 + x_2 = 0 \\ 2x_1 - x_2 = 0 \end{cases}$$

Inhomogeneous System.

$$\begin{cases} x_1 + x_2 = 3 \\ 2x_1 - x_2 = 0 \end{cases}$$