opHW11

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Sec 5.3, 11*.

来自 <https://users.math.msu.edu/users/zhangshiwen/s19/homework.html>

11. Let $P = A(A^TA)^{-1}A^T$, where A is an $m \times n$ matrix of rank n.

(a) Show that $P^2 = P$.

(b) Prove that $P^k = P$ for $k = 1, 2, \ldots$.

(c) Show that P is symmetric. [*Hint*: If B is nonsingular, then $(B^{-1})^T = (B^T)^{-1}$.]

(a)

$$P^2 = P \cdot P = A \cdot \underbrace{(A^T A)^{-1} \cdot \overbrace{A^T \cdot A}}(A^T A)^{-1} \cdot A^T$$

$$= A \cdot I \cdot (A^T A)^{-1} \cdot A^T$$

$$= P.$$

(b) by (a)

$$P^3 = P^2 \cdot P = P \cdot P = P.$$

inductively, for all $k \geq 3$

$$P^k = P^2 \cdot P^{k-2} = P \cdot P^{k-2} = P^{k-1} = P^{k-2} = \cdots = P$$

$$P^k = P^2 \cdot P^{k-2} = P \cdot P^{k-2} = P^{k-1} = P^{k-2} = \cdots = P.$$

(c)

$$P^T = \left(A (A^T A)^{-1} \cdot A^T \right)^T$$

$$= (A^T)^T \cdot \left((A^T A)^{-1} \right)^T \cdot A^T$$

$$= A \cdot \left((A^T A)^T \right)^{-1} \cdot A^T$$

$$= A (A^T A)^{-1} \cdot A^T$$

$$= P.$$

Sec 5.4, 21*, 23*, 33*

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21. Let $\mathbf{x} \in \mathbb{R}^n$. Show that $\|\mathbf{x}\|_\infty \leq \|\mathbf{x}\|_2$.

$$\bar{x} = (x_1, x_2, \ldots, x_n)^T$$

$$\|\bar{x}\|_\infty = \max_{1 \leq i \leq n} |x_i|$$

$$\|\bar{x}\|_2 = \sqrt{|x_1|^2 + \cdots + |x_n|^2}$$

Suppose for some $1 \le j \le n$,

$|x_j|$ is the largest among all $|x_1|, |x_2|, \cdots, |x_n|$

Then

$$\|\mathbf{x}\|_2 = \sqrt{|x_1|^2 + \cdots + |x_j|^2 + \cdots + |x_n|^2}$$

$$\ge \sqrt{|x_j|^2}$$

$$= |x_j| = \|\mathbf{x}\|_\infty$$

23. Give an example of a nonzero vector $\mathbf{x} \in \mathbb{R}^2$ for which

$$\|\mathbf{x}\|_\infty = \|\mathbf{x}\|_2 = \|\mathbf{x}\|_1$$

Let $\mathbf{x} = (1, 0)^T$

Then $\|\mathbf{x}\|_\infty = 1$

$\|\mathbf{x}\|_2 = \sqrt{1^2 + 0^2} = 1$

$\|\mathbf{x}\|_1 = |1| + |0| = 1$.

33. Consider the vector space $\mathbb{R}^n$ with inner product $\langle \mathbf{x}, \mathbf{y} \rangle = \mathbf{x}^T\mathbf{y}$. Show that for any $n \times n$ matrix A,

(a) $\langle A\mathbf{x}, \mathbf{y} \rangle = \langle \mathbf{x}, A^T\mathbf{y} \rangle$

(b) $\langle A^TA\mathbf{x}, \mathbf{x} \rangle = \|A\mathbf{x}\|^2$

(a)

$$\langle A\bar{x}, \bar{y} \rangle = (A\bar{x})^T \cdot \bar{y}$$

$$= (\bar{x}^T \cdot A^T) \cdot \bar{y}$$

$$= \bar{x}^T \cdot (A^T \cdot \bar{y})$$

$$= \langle \bar{x}, A^T\bar{y} \rangle.$$

(b)

$$\langle A^T \cdot A\bar{x}, \bar{x} \rangle = (\underline{A^T} \cdot \underline{A\bar{x}})^T \cdot \bar{x}$$

$$= (A\bar{x})^T \cdot (A^T)^T \cdot \bar{x}$$

$$= (A\bar{x})^T \cdot (A\bar{x}).$$

$$= \langle A\bar{x}, A\bar{x} \rangle$$

$$= \|A\underline{x}\|^2$$