HW9

2019年3月27日 11:03

Sec 6.3, 1(e), 2 (a), 5

来自 <https://users.math.msu.edu/users/zhangshiwen/s19/homework.html>

1. In each of the following, factor the matrix A into a product XDX^{-1}, where D is diagonal:

(e) $A = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 3 \\ 1 & 1 & -1 \end{bmatrix}$

$$p(\lambda) = \begin{vmatrix} 1-\lambda & 0 & 0 \\ -2 & 1-\lambda & 3 \\ 1 & 1 & -1-\lambda \end{vmatrix}$$

$$= (1-\lambda)\cdot\left[(1-\lambda)\cdot(-1-\lambda) - 3\right]$$

$$= (1-\lambda)\cdot\left[\lambda^2 - 4\right]$$

$$= (1-\lambda)\cdot(\lambda-2)(\lambda+2) = 0$$

$\lambda_1 = 1$, $\lambda_2 = 2$, $\lambda_3 = -2$.

$$A - I = \begin{bmatrix} 0 & 0 & 0 \\ -2 & 0 & 3 \\ 1 & 1 & -2 \end{bmatrix}$$

$$\longrightarrow \begin{bmatrix} 1 & 1 & -2 \\ 0 & 2 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\begin{cases} x_1 + x_2 - 2x_3 = 0 \\ 2x_2 - x_3 = 0 \end{cases}$$

$x_3 = 2 \Rightarrow x_2 = 1, \ x_1 = 3$

e-vector of $\lambda_1 = 1$. $v_1 = \begin{pmatrix} 3 \\ 1 \\ 2 \end{pmatrix}$

$\lambda_2 = 2$

$$A - 2I = \begin{bmatrix} -1 & 0 & 0 \\ -2 & -1 & 3 \\ 1 & 1 & -3 \end{bmatrix}$$

$$\rightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 3 \\ 0 & 1 & -3 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -3 \\ 0 & 0 & 0 \end{bmatrix}.$$

$$v_2 = \begin{bmatrix} 0 \\ 3 \\ 1 \end{bmatrix}$$

$\lambda_3 = -2$

$$A + 2I = \begin{bmatrix} 3 & 0 & 0 \\ -2 & 3 & 3 \\ 1 & 1 & 1 \end{bmatrix}$$

$$\to \begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & 3 \\ 0 & 1 & 1 \end{bmatrix}$$

$$\to \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

$$v_3 = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$$

$$X = (v_1, v_2, v_3) = \begin{bmatrix} 3 & 0 & 1 \\ 1 & 3 & 1 \\ 2 & 1 & -1 \end{bmatrix}, \quad D = \begin{bmatrix} 1 & & \\ & 2 & \\ & & -2 \end{bmatrix}$$

Then $X^{-1} A X = D$

and $A = X \cdot D X^{-1}$.

2. For each of the matrices in Exercise 1, use the XDX^{-1} factorization to compute A^6.

(a) $A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

By 1(a).

$$A = X \cdot D X^{-1} = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \cdot \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{pmatrix}$$

Then

$$A^6 = X \cdot D^6 \cdot X^{-1}$$

$$= X \cdot \begin{bmatrix} 1^6 & \\ & (-1)^6 \end{bmatrix} \cdot X^{-1} = X \cdot I \cdot X^{-1} = I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

5. Let A be a nondefective $n \times n$ matrix with diagonalizing matrix X. Show that the matrix $Y = (X^{-1})^T$ diagonalizes A^T.

If $X^{-1} A X = D$

then.

$$(X^{-1}AX)^T = D^T$$

$$\Rightarrow X^T \cdot A^T \cdot (X^{-1})^T = D^T = D$$

Notice $X^T = ((X^{-1})^T)^{-1}$

Therefore, $(X^{-1})^T$ diagonalizes A^T.

Sec4.1, 1(a)(d)(e), 3, 4, 5(b)(c), 6(a)(c), 9(a)(b), 17(a)

来自 <https://users.math.msu.edu/users/zhangshiwen/s19/homework.html>

1. Show that each of the following are linear operators on $\mathbb{R}^2$. Describe geometrically what each linear transformation accomplishes.

(a) $L(\mathbf{x}) = (-x_1, x_2)^T$ **(b)** $L(\mathbf{x}) = -\mathbf{x}$

(c) $L(\mathbf{x}) = (x_2, x_1)^T$ **(d)** $L(\mathbf{x}) = \frac{1}{2}\mathbf{x}$

(e) $L(\mathbf{x}) = x_2\mathbf{e}_2$

(a) For $\bar{x}, \bar{y} \in \mathbb{R}^2$, $\alpha \in \mathbb{R}$.

$$L(\bar{x}+\bar{y}) = L\left(\begin{pmatrix} x_1+y_1 \\ x_2+y_2 \end{pmatrix}\right)$$

$$= \begin{pmatrix} -(x_1+y_1) \\ x_2+y_2 \end{pmatrix}$$

$$\begin{pmatrix} \\ x_2+y_2 \end{pmatrix}$$

$$= \begin{pmatrix} -x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} -y_1 \\ y_2 \end{pmatrix} = L(\bar{x}) + L(\bar{y})$$

$$L(\alpha \bar{x}) = L\left(\begin{pmatrix} \alpha x_1 \\ \alpha x_2 \end{pmatrix}\right) = \begin{pmatrix} -(\alpha \cdot x_1) \\ \alpha x_2 \end{pmatrix}$$

$$= \alpha \cdot \begin{pmatrix} -x_1 \\ x_2 \end{pmatrix} = \alpha \cdot L(\bar{x}).$$

(d) $L(\bar{x}+\bar{y}) = \frac{1}{2}(\bar{x}+\bar{y}) = \frac{1}{2}\bar{x} + \frac{1}{2}\bar{y}$

$$= L(\bar{x}) + L(\bar{y})$$

$$L(\alpha \cdot \bar{x}) = \frac{1}{2}(\alpha \bar{x}) = \alpha \cdot (\tfrac{1}{2}\bar{x})$$

$$= \alpha \cdot L(\bar{x})$$

(e) $L(\bar{x}+\bar{y}) = (x_2+y_2) \cdot \bar{e_2}$

$$= x_2 \cdot \bar{e_2} + y_2 \cdot \bar{e_2}$$

$$= L(\bar{x}) + L(\bar{y})$$

$= L(\bar{x}) + L(\bar{y})$

$L(\alpha \cdot \bar{x}) = (\alpha x_2) \cdot \bar{e_2}$

$= \alpha \cdot (x_2 \cdot \bar{e_2}) = \alpha \cdot L(\bar{x})$.

3. Let **a** be a fixed nonzero vector in $\mathbb{R}^2$. A mapping of the form

$$L(\mathbf{x}) = \mathbf{x} + \mathbf{a}$$

is called a *translation*. Show that a translation is not a linear operator. Illustrate geometrically the effect of a translation.

$\bar{0} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \in \mathbb{R}^2$

$L(\bar{0}) = \bar{0} + \bar{a} = \bar{a} \neq \bar{0}$

Therefore, L is not a linear operator.

4. Let $L: \mathbb{R}^2 \to \mathbb{R}^2$ be a linear operator. If

$$L((1, 2)^T) = (-2, 3)^T$$

and

$$L((1, -1)^T) = (5, 2)^T$$

find the value of $L((7, 5)^T)$.

Suppose

$$\begin{pmatrix}7\\5\end{pmatrix} = c_1\cdot\begin{pmatrix}1\\2\end{pmatrix} + c_2\begin{pmatrix}1\\-1\end{pmatrix}$$

$$\Leftrightarrow \begin{bmatrix}1 & 1\\2 & -1\end{bmatrix}\cdot\begin{bmatrix}c_1\\c_2\end{bmatrix} = \begin{bmatrix}7\\5\end{bmatrix}$$

$$\left[\begin{array}{cc|c}1 & 1 & 7\\2 & -1 & 5\end{array}\right]$$

$$\to \left[\begin{array}{cc|c}1 & 1 & 7\\0 & -3 & -9\end{array}\right] \Rightarrow \begin{array}{r}x_1+x_2=7\\-3x_2=-9\end{array}$$

$$\begin{cases}x_1=4\\x_2=3\end{cases}$$

$$L\left(\begin{pmatrix}7\\5\end{pmatrix}\right) = L\left(4\cdot\begin{pmatrix}1\\2\end{pmatrix}+3\begin{pmatrix}1\\-1\end{pmatrix}\right)$$

$$= 4\cdot L\begin{pmatrix}1\\2\end{pmatrix} + 3\cdot L\left(\begin{pmatrix}1\\-1\end{pmatrix}\right)$$

$$= 4\cdot\begin{pmatrix}-2\\3\end{pmatrix} + 3\cdot\begin{pmatrix}5\\2\end{pmatrix}$$

$$= \begin{pmatrix}7\\18\end{pmatrix}$$

5. Determine whether the following are linear transformations from $\mathbb{R}^3$ into $\mathbb{R}^2$.

(a) $L(\mathbf{x}) = (x_2, x_3)^T$ **(b)** $L(\mathbf{x}) = (0, 0)^T$

(c) $L(\mathbf{x}) = (1 + x_1, x_2)^T$

(b) Yes.

$$L(\bar{x}+\bar{y}) = \begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \end{pmatrix} = L(\bar{x}) + L(\bar{y})$$

$$L(\alpha \cdot \bar{x}) = \begin{pmatrix} 0 \\ 0 \end{pmatrix} = \alpha \cdot \begin{pmatrix} 0 \\ 0 \end{pmatrix} = \alpha \cdot L(\bar{x}).$$

(c). No. For $\bar{0} \in \mathbb{R}^3$

$$L(\bar{0}) = \begin{pmatrix} 1+0 \\ 0 \end{pmatrix} \neq \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

6. Determine whether the following are linear transformations from $\mathbb{R}^2$ into $\mathbb{R}^3$.

(a) $L(\mathbf{x}) = (x_1, x_2, 1)^T$

(a) No. For $\bar{0} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \in \mathbb{R}^2$,

$$L(\bar{0}) = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \neq \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

(c) $L(\mathbf{x}) = (x_1, 0, 0)^T$

Yes.

$\overline{x}, \overline{y} \in \mathbb{R}^2$.

$$\overline{x}+\overline{y} = \begin{pmatrix} x_1+y_1 \\ x_2+y_2 \end{pmatrix}$$

$$L(\overline{x}+\overline{y}) = \begin{pmatrix} x_1+y_1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} x_1 \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} y_1 \\ 0 \\ 0 \end{pmatrix}$$

$$= L(\overline{x}) + L(\overline{y}).$$

$$L(\alpha \cdot \overline{x}) = \begin{pmatrix} \alpha x_1 \\ 0 \\ 0 \end{pmatrix} = \alpha \cdot \begin{pmatrix} x_1 \\ 0 \\ 0 \end{pmatrix} = \alpha \cdot L(\overline{x}).$$

9. Determine whether the following are linear transformations from P_2 to P_3.

(a) $L(p(x)) = xp(x)$

(b) $L(p(x)) = x^2 + p(x)$

(a) Yes.

For $p(x), q(x) \in P_3$

$$L(p(x)+q(x)) = x(p(x)+q(x))$$

$= x \cdot p(x) + x \cdot q(x)$

$= L(p(x)) + L(q(x))$

$L(\alpha \cdot p(x)) = x \cdot (\alpha p(x))$

$= \alpha \cdot (x \cdot p(x)) = \alpha \cdot L(p(x))$

(b) No.

$p(x) = 1 \in P_3$. let $\alpha = 2$

$L(\alpha \cdot p(x)) = x^2 + \alpha p(x) = x^2 + 2$

while $\alpha \cdot L(p(x)) = 2 \cdot [x^2 + 1] \neq x^2 + 2$.

17. Determine the kernel and range of each of the following linear operators on $\mathbb{R}^3$:

(a) $L(\mathbf{x}) = (x_3, x_2, x_1)^T$ **(b)** $L(\mathbf{x}) = (x_1, x_2, 0)^T$

$L(\bar{x}) = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \Leftrightarrow \begin{pmatrix} x_3 \\ x_2 \\ x_1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$

$\Rightarrow x_1 = x_2 = x_3 = 0$

$\ker(L) = \{ \begin{pmatrix} 0 \\ 0 \\ \end{pmatrix} \}$

$\ker(L) = \left\{ \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \right\}$

Sec4.2, 2(a)(b), 3(c), 4 (a)

来自 <https://users.math.msu.edu/users/zhangshiwen/s19/homework.html>

2. For each of the following linear transformations L mapping $\mathbb{R}^3$ into $\mathbb{R}^2$, find a matrix A such that $L(\mathbf{x}) = A\mathbf{x}$ for every $\mathbf{x}$ in $\mathbb{R}^3$:

(a) $L((x_1, x_2, x_3)^T) = (x_1 + x_2, 0)^T$

(b) $L((x_1, x_2, x_3)^T) = (x_1, x_2)^T$

(a). $\bar{e_1} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$, $\bar{e_2} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$, $\bar{e_3} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$

$L(\bar{e_1}) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, $L(\bar{e_2}) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, $L(\bar{e_3}) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$.

Matrix representation

$$A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

(b).

$L(\bar{e_1}) = \begin{pmatrix} 1 \\ \end{pmatrix}$ $L(\bar{e_2}) = \begin{pmatrix} 0 \\ \end{pmatrix}$ $L(\bar{e_3}) = \begin{pmatrix} 0 \\ \end{pmatrix}$

$L(\bar{e}_1)=\begin{pmatrix}1\\0\end{pmatrix}$, $L(\bar{e}_2)=\begin{pmatrix}0\\1\end{pmatrix}$, $L(\bar{e}_3)=\begin{pmatrix}0\\0\end{pmatrix}$.

Matrix representation

$$A=\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\end{bmatrix}.$$

3. For each of the following linear operators L on $\mathbb{R}^3$, find a matrix A such that $L(\mathbf{x}) = A\mathbf{x}$ for every $\mathbf{x}$ in $\mathbb{R}^3$:

(a) $L((x_1, x_2, x_3)^T) = (x_3, x_2, x_1)^T$

(b) $L((x_1, x_2, x_3)^T) = (x_1, x_1 + x_2, x_1 + x_2 + x_3)^T$

(c) $L((x_1, x_2, x_3)^T) = (2x_3, x_2 + 3x_1, 2x_1 - x_3)^T$

(c) $L(\bar{e}_1)=\begin{bmatrix}0\\3\\2\end{bmatrix}$, $L(e_2)=L\begin{pmatrix}0\\1\\0\end{pmatrix}=\begin{pmatrix}0\\1\\0\end{pmatrix}$

$L(\bar{e}_3)=L\begin{pmatrix}0\\0\\1\end{pmatrix}=\begin{pmatrix}2\\0\\-1\end{pmatrix}$.

Matrix representation

$$A=\begin{bmatrix}0 & 0 & 2\\ 3 & 1 & 0\\ 2 & 0 & -1\end{bmatrix}$$

4. Let L be the linear operator on $\mathbb{R}^3$ defined by

$$L(\mathbf{x}) = \begin{bmatrix} 2x_1 - x_2 - x_3 \\ 2x_2 - x_1 - x_3 \\ 2x_3 - x_1 - x_2 \end{bmatrix}$$

Determine the standard matrix representation A of L, and use A to find $L(\mathbf{x})$ for each of the following vectors $\mathbf{x}$:

(a) $\mathbf{x} = (1, 1, 1)^T$ **(b)** $\mathbf{x} = (2, 1, 1)^T$

(a)

$L(\bar{e}_1) = L\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{bmatrix} 2 \\ -1 \\ -1 \end{bmatrix}$, $L(\bar{e}_2) = L\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} -1 \\ 2 \\ -1 \end{pmatrix}$

$L(\bar{e}_3) = L\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{bmatrix} -1 \\ -1 \\ 2 \end{bmatrix}$

Matrix representation

$$A = \begin{bmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{bmatrix}$$

$$L(\bar{x}) = L\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = A\bar{x} = \begin{bmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$