HW8

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- **Sec 6.1, 1(a)(c)(f)(h), 2, 3, 4, 14.**
- **Sec 6.3, 1(a)(c)**

来自 <https://users.math.msu.edu/users/zhangshiwen/s19/homework.html>

27

1. Find the eigenvalues and the corresponding eigenspaces for each of the following matrices:

6 **(a)** $\begin{bmatrix} 3 & 2 \\ 4 & 1 \end{bmatrix}$ **(b)** $\begin{bmatrix} 6 & -4 \\ 3 & -1 \end{bmatrix}$

6 **(c)** $\begin{bmatrix} 3 & -1 \\ 1 & 1 \end{bmatrix}$ **(d)** $\begin{bmatrix} 3 & -8 \\ 2 & 3 \end{bmatrix}$

(e) $\begin{bmatrix} 1 & 1 \\ -2 & 3 \end{bmatrix}$ 6 **(f)** $\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$

(g) $\begin{bmatrix} 1 & 1 & 1 \\ 0 & 2 & 1 \\ 0 & 0 & 1 \end{bmatrix}$ 9 **(h)** $\begin{bmatrix} 1 & 2 & 1 \\ 0 & 3 & 1 \\ 0 & 5 & -1 \end{bmatrix}$

6pt (a).

$$p(\lambda) = \begin{vmatrix} 3-\lambda & 2 \\ 4 & 1-\lambda \end{vmatrix}$$

$$= (3-\lambda)(1-\lambda) - 8$$

$$= \lambda^2 - 4\lambda - 5$$

$$= (\lambda-5)(\lambda+1) = 0$$

Eigenvalues: $\lambda_1 = 5$, $\lambda_2 = -1$.

1pt 1pt

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1pt/6 1pt/6

$\lambda_1 = 5$.

$$A - 5I = \begin{bmatrix} -2 & 2 \\ 4 & -4 \end{bmatrix} \to \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix},$$

$\Rightarrow x_1 = x_2$, $x_2 = \alpha$.

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \alpha \cdot \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

eigenspace of $\lambda_1 = 5$

$$N(A - 5I) = \left\{ \alpha \begin{pmatrix} 1 \\ 1 \end{pmatrix} \,\middle|\, \alpha \in \mathbb{R} \right\}.$$ 2pt/6

$\lambda_2 = -1$.

$$A + I = \begin{bmatrix} 4 & 2 \\ 4 & 2 \end{bmatrix} \to \begin{bmatrix} 2 & 1 \\ 0 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & \frac{1}{2} \\ 0 & 0 \end{bmatrix}$$

$x_1 = -\frac{1}{2} x_2$, $x_2 = \alpha$. $\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \alpha \begin{pmatrix} -\frac{1}{2} \\ 1 \end{pmatrix}$

eigenspace of $\lambda_2 = -1$ $\left(\text{or } \alpha \cdot \begin{pmatrix} -1 \\ 2 \end{pmatrix} \right)$

$$N(A + I) = \left\{ \alpha \cdot \begin{pmatrix} -\frac{1}{2} \\ 1 \end{pmatrix} \,\middle|\, \alpha \in \mathbb{R} \right\}$$ 2pt/6

$$\left(\text{or } \left\{ \alpha \cdot \begin{pmatrix} -1 \\ 2 \end{pmatrix} \,\middle|\, \alpha \in \mathbb{R} \right\} \right)$$

6pt (c).

6pt (c).

$$p(\lambda) = \begin{vmatrix} 3-\lambda & -1 \\ 1 & 1-\lambda \end{vmatrix}$$

$$= (3-\lambda)(1-\lambda)+1$$

$$= \lambda^2 - 4\lambda + 2$$

$$= (\lambda-2)^2$$ 2pt/6

eigenvalue $\lambda = 2$ 2pt/6

(or $\lambda_1 = \lambda_2 = 2$)

$\lambda = 2$.

$$A - 2I = \begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}$$

$x_1 = x_2$, $x_2 = \alpha$

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \alpha \cdot \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

eigenspace of $\lambda = 2$ is

$$N(A-2I) = \left\{ \alpha \begin{pmatrix} 1 \\ 1 \end{pmatrix} \middle| \alpha \in \mathbb{R} \right\}$$ 2pt/6

6pt (f)

$$p(\lambda) = \begin{vmatrix} -\lambda & 1 & 0 \\ 0 & -\lambda & 1 \\ \ldots & & \ldots \end{vmatrix}$$

$$p(\lambda) = \begin{vmatrix} 0 & -\lambda & 1 \\ 0 & 0 & -\lambda \end{vmatrix}$$

$$= (-\lambda)^3 = 0 \qquad 2pt/6$$

eigenvalue $\lambda = 0$ 2pt/6

(or $\lambda_1 = \lambda_2 = \lambda_3 = 0$)

$$A - 0 \cdot I = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \quad \text{(this is R.R.E.F.)}$$

$$\begin{cases} x_2 = 0 \\ x_3 = 0 \end{cases} \text{(lead).} \qquad x_1 = \alpha \text{ (free).}$$

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \alpha \cdot \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}.$$

eigenspace of $\lambda = 0$ is

$$N(A) = \left\{ \alpha \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \,\middle|\, \alpha \in \mathbb{R} \right\}. \qquad 2pt/6$$

9pt (h)

See (h) in the very end of the file.

4pt 2.

Let $A = \begin{bmatrix} a_{11} & * & \cdots & * \\ 0 & a_{22} & \ddots & \vdots \\ \vdots & \ddots & \ddots & * \\ 0 & \cdots & 0 & a_{nn} \end{bmatrix}$ be an upper triangular matrix.

$f(\lambda) = \det(A - \lambda I) = \begin{vmatrix} a_{11}-\lambda & \cdots \end{vmatrix}$

$$p(\lambda)=\det(A-\lambda I)=\begin{vmatrix} a_{11}-\lambda & \cdots & & \vdots \\ & a_{22}-\lambda & & \vdots \\ & & \ddots & \vdots \\ & & & a_{nn}-\lambda \end{vmatrix}$$

$$=(a_{11}-\lambda)(a_{22}-\lambda)\cdots\cdots(a_{nn}-\lambda)=0$$ 2pt/4

This implies the eigenvalues of A are

$\lambda_1=a_{11}$, $\lambda_2=a_{22}$, $\ldots$, $\lambda_n=a_{nn}$. 2pt/4

2. Show that the eigenvalues of a triangular matrix are the diagonal elements of the matrix.

3. Let A be an $n \times n$ matrix. Prove that A is singular if and only if $\lambda = 0$ is an eigenvalue of A.

4pt Proof 1.

Let $\lambda_1, \lambda_2, \ldots, \lambda_n$ be the eigenvalues of A.

Then $\det A = \lambda_1 \cdots \lambda_n$ 2pt/4

If $\lambda=0$ is an eigenvalue, then one of $\lambda_1, \ldots, \lambda_n$ is zero. Therefore,

$$\det A = 0,$$ 2pt/4.

which implies A is singular

If A is singular, then $\lambda_1 \cdots \lambda_n = 0$. one of them must be 0.

or 4pt Proof 2.

or 4pt Proof 2.

$\lambda = 0$ is an eigenvalue

By the definition of evalue,

there is a nonzero vector $\bar{x} \in \mathbb{R}^n$ such that

$$A\bar{x} = \lambda\bar{x} = \bar{0}$$ 2pt/4

This means $A\bar{x} = \bar{0}$ has a nontrivial solution. 2pt/4.

Therefore, A is singular.

5pt **4.** Let A be a nonsingular matrix and let λ be an eigenvalue of A. Show that $1/\lambda$ is an eigenvalue of A^{-1}.

Pf. If λ is an eigenvalue of A, then

there is a non-zero vector $\bar{x}$ such that

$$A\bar{x} = \lambda\bar{x}$$ 2pt/5

$$\Rightarrow \quad A^{-1}(A\bar{x}) = A^{-1} \cdot \lambda \cdot \bar{x}$$

$$\Rightarrow \quad \bar{x} = \lambda \cdot (A^{-1}\bar{x}).$$

$$\Rightarrow \quad \lambda^{-1} \cdot \bar{x} = \lambda^{-1} \cdot \lambda \cdot (A^{-1}\bar{x}) \qquad (\text{by } \#3,\ \lambda \neq 0).$$

$$\Rightarrow \quad \lambda^{-1}\bar{x} = A^{-1} \cdot \bar{x}.$$

$\Rightarrow \quad \lambda^{-1} \bar{x} = A^{-1} \cdot \bar{x}.$

$\Leftrightarrow \quad A^{-1} \bar{x} = \lambda^{-1} \bar{x}.$ 3pt/5.

i.e. λ^{-1} is an eigenvalue of A^{-1}.

4pt **14.** Let A be a 2×2 matrix. If $\operatorname{tr}(A) = 8$ and $\det(A) = 12$, what are the eigenvalues of A?

Let λ_1, λ_2 be the eigenvalues of A

Then $\lambda_1 + \lambda_2 = \operatorname{tr}(A) = 8$ 1pt/4

and $\lambda_1 \cdot \lambda_2 = \det(A) = 12$ 1pt/4

The solution is:

$\lambda_1 = 2. \quad \lambda_2 = 6$

1pt/4 1pt/4.

1. In each of the following, factor the matrix A into a product XDX^{-1}, where D is diagonal:

(a) $A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ **(b)** $A = \begin{pmatrix} 5 & 6 \\ -2 & -2 \end{pmatrix}$

(c) $A = \begin{pmatrix} 2 & -8 \\ 1 & -4 \end{pmatrix}$ **(d)** $A = \begin{pmatrix} 2 & 2 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & -1 \end{pmatrix}$

8pt (a).

$$p(\lambda) = \begin{vmatrix} -\lambda & 1 \\ 1 & -\lambda \end{vmatrix} = \lambda^2 - 1 = 0$$

$\lambda_1 = 1$, $\lambda_2 = -1$. eigenvalues.

1pt/8 1pt/8

$$A - I = \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix} \to \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}, \quad x_1 - x_2 = 0$$

$\Rightarrow v_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ is an eigenvector corresponding to $\lambda_1 = 1$.

1pt/8

$$A + I = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}, \quad x_1 + x_2 = 0$$

$\Rightarrow v_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$ is an eigenvector of $\lambda_2 = -1$.

1pt/8

let $X = (v_1, v_2) = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$, $X^{-1} = \frac{1}{-2} \begin{bmatrix} -1 & -1 \\ -1 & 1 \end{bmatrix}$

1pt/8 1pt/8

and

$$D = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

1pt/8

Then

$$A = X \cdot D X^{-1} = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \cdot \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \cdot & \cdot \end{bmatrix}$$

$$A = X \cdot D X^{-1} = \underbrace{\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \cdot \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{pmatrix}}_{1pt/8}$$

8pt (c).

$$p(\lambda) = \begin{vmatrix} 2-\lambda & -8 \\ 1 & -4-\lambda \end{vmatrix} = (2-\lambda)(-4-\lambda) + 8$$

$$= \lambda^2 + 2\lambda$$

$$= \lambda(\lambda+2) = 0$$

$\lambda_1 = 0$, $\lambda_2 = -2$.

1pt/8 1pt/8

$$A - 0 \cdot I = \begin{bmatrix} 2 & -8 \\ 1 & -4 \end{bmatrix} \to \begin{bmatrix} 1 & -4 \\ 0 & 0 \end{bmatrix}, \quad x_1 = 4x_2$$

$v_1 = \begin{pmatrix} 4 \\ 1 \end{pmatrix}$ is an e-vector of $\lambda_1 = 0$.

1pt/8

$$A + 2 \cdot I = \begin{bmatrix} 4 & -8 \\ 1 & -2 \end{bmatrix} \to \begin{bmatrix} 1 & -2 \\ 0 & 0 \end{bmatrix}, \quad x_1 = 2x_2$$

$v_2 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$ is an e-vector of $\lambda_2 = -2$.

1pt/8

Let $X = (v_1, v_2) = \begin{bmatrix} 4 & 2 \\ 1 & 1 \end{bmatrix}$ 1pt/8, $X^{-1} = \frac{1}{2} \cdot \begin{bmatrix} 1 & -2 \\ -1 & 4 \end{bmatrix}$ 1pt/8

$D = \begin{bmatrix} \lambda_1 & \end{bmatrix} \begin{bmatrix} 0 & 0 \end{bmatrix}$

1/8

$$D = \begin{bmatrix} \lambda_1 & \\ & \lambda_2 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & -2 \end{bmatrix}$$

1pt/8

Then

$$A = X \cdot D \cdot X^{-1} = \begin{bmatrix} 4 & 2 \\ 1 & 1 \end{bmatrix} \cdot \underbrace{\begin{bmatrix} 0 & 0 \\ 0 & 2 \end{bmatrix} \cdot \begin{bmatrix} \frac{1}{2} & -1 \\ -\frac{1}{2} & 2 \end{bmatrix}}_{1pt/8}$$

§6.1

1. (h).

9pt (h)

$$p(\lambda) = \begin{vmatrix} 1-\lambda & 2 & 1 \\ 0 & 3-\lambda & 1 \\ 0 & 5 & -1-\lambda \end{vmatrix}$$

$$= (1-\lambda) \cdot \begin{vmatrix} 3-\lambda & 1 \\ 5 & -1-\lambda \end{vmatrix}$$

$$= (1-\lambda) \cdot \left((3-\lambda)(-1-\lambda) - 5 \right)$$

$$= (1-\lambda)(\lambda^2 - 2\lambda - 8)$$

$$= (1-\lambda) \cdot (\lambda - 4)(\lambda + 2) = 0$$

eigenvalues

eigenvalues

$\lambda_1=1, \quad \lambda_2=4, \quad \lambda_2=-2$

1pt/9 1pt/9 1pt/9

$\lambda_1=1$

$$A-I=\begin{bmatrix}0&2&1\\0&2&1\\0&5&-2\end{bmatrix}\to\begin{bmatrix}0&2&1\\0&5&-2\\0&0&9\end{bmatrix}.$$

$$\begin{cases}2x_2+x_3=0\\5x_2-2x_3=0\end{cases}\Rightarrow x_2=x_3=0 \text{ (lead)}, \quad x_1=\alpha \text{ (free)}$$

$$\begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix}=\alpha\cdot\begin{pmatrix}1\\0\\0\end{pmatrix}.$$

eigenspace of $\lambda_1=1$

$$N(A-I)=\left\{\alpha\cdot\begin{pmatrix}1\\0\\0\end{pmatrix} \middle| \alpha\in\mathbb{R}\right\}$$

2pt/9

$\lambda_2=4$

$$A-4I=\begin{bmatrix}-3&2&1\\0&-1&1\\0&5&-5\end{bmatrix}$$

$$\to\begin{bmatrix}-3&2&1\\0&1&-1\\0&0&0\end{bmatrix}$$

$$\to\begin{bmatrix}-3&0&3\\0&1&-1\end{bmatrix}\to\begin{bmatrix}1&0&-1\\0&1&-1\end{bmatrix}$$

$$\rightarrow \begin{bmatrix} [illegible] & [illegible] & [illegible] \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} [illegible] & [illegible] & [illegible] \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\begin{cases} x_1 = x_3 \\ x_2 = x_3 \end{cases} \qquad x_3 = \alpha$$

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \alpha \cdot \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}.$$

eigenspace of $\lambda_2 = 4$.

$$N(A-4I) = \left\{ \alpha \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \middle| \alpha \in \mathbb{R} \right\}.$$

2pt/9

$$\lambda_3 = -2$$

$$A+2I = \begin{bmatrix} 3 & 2 & 1 \\ 0 & 5 & 1 \\ 0 & 5 & 1 \end{bmatrix}$$

$$\rightarrow \begin{bmatrix} 3 & 2 & 1 \\ 0 & 5 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\rightarrow \begin{bmatrix} 3 & 0 & \frac{3}{5} \\ 0 & 5 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\begin{cases} x_1 = -\frac{1}{5} x_3 \\ x_2 = -\frac{1}{5} x_3 \end{cases}, \quad x_3 = \alpha .$$

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \alpha \cdot \begin{pmatrix} -\frac{1}{5} \\ -\frac{1}{5} \\ 1 \end{pmatrix} \quad \text{or} \quad \alpha \cdot \begin{pmatrix} -1 \\ -1 \\ 5 \end{pmatrix}$$

2pt /9.

e-space of $\lambda_3 = -2$:

$$N(A+2I) = \left\{ \alpha \cdot \begin{pmatrix} -1 \\ -1 \\ 5 \end{pmatrix} \middle| \alpha \in \mathbb{R} \right\}$$