HW7

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- **Sec 3.3, 1(a)(b)(c), 2 (a) (c) (e), 8 (a) (b).**
- **Sec 3.4, 3, 4, 5, 14 (a)(b)(c)(d).**
- **Sec 3.5, 1 (a), 3 (b)**

来自 <https://users.math.msu.edu/users/zhangshiwen/s19/homework.html>

7pt **1.** Determine whether the following vectors are linearly independent in $\mathbb{R}^2$:

(a) $\begin{bmatrix} 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 3 \\ 2 \end{bmatrix}$ **(b)** $\begin{bmatrix} 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 4 \\ 6 \end{bmatrix}$

(c) $\begin{bmatrix} -2 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 3 \end{bmatrix}, \begin{bmatrix} 2 \\ 4 \end{bmatrix}$

(a) 2pt $X = \begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix}$

$\det X = 2\cdot 2 - 3\cdot 1 = 1 \neq 0$ 2pt/2

Therefore, $\begin{bmatrix} 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 3 \\ 2 \end{bmatrix}$ are linearly independent.

(b) 2pt $X = \begin{bmatrix} 2 & 4 \\ 3 & 6 \end{bmatrix}$

$\det X = 2\cdot 6 - 3\cdot 4 = 0$

$\begin{bmatrix} 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 4 \\ 6 \end{bmatrix}$ are linearly dependent 2pt/2.

Or: $\begin{bmatrix} 4 \\ 6 \end{bmatrix} = 2\cdot\begin{bmatrix} 2 \\ 3 \end{bmatrix}$. $\Rightarrow$ linearly dependent

(c) 3pt: $c_1 \begin{bmatrix} -2 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ 3 \end{bmatrix} + c_3 \begin{bmatrix} 2 \\ 4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ 1pt/3

Augmented matrix:

$$\left[\begin{array}{ccc:c} -2 & 1 & 2 & 0 \\ 1 & 3 & 4 & 0 \end{array}\right]$$

$$\rightarrow \left[\begin{array}{ccc:c} 1 & 3 & 4 & 0 \\ 0 & 7 & 10 & 0 \end{array}\right]$$

Therefore, the linear system has one free variables c_3. The system has nontrivial solution: 2pt/3

The three vectors are linearly dependent.

$\boxed{\text{Or}}$: More than 2 vectors in $\mathbb{R}^2$ are linearly dependent. (this gets full credits)

7pt **2.** Determine whether the following vectors are linearly independent in $\mathbb{R}^3$:

(a) $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$

(b) $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$

(c) $\begin{bmatrix} 2 \\ 1 \\ -2 \end{bmatrix}, \begin{bmatrix} 3 \\ 2 \\ -2 \end{bmatrix}, \begin{bmatrix} 2 \\ 2 \\ 0 \end{bmatrix}$

(d) $\begin{bmatrix} 2 \\ 1 \\ -2 \end{bmatrix}, \begin{bmatrix} -2 \\ -1 \\ 2 \end{bmatrix}, \begin{bmatrix} 4 \\ 2 \\ -4 \end{bmatrix}$

(e) $\begin{bmatrix} 1 \\ 1 \\ \end{bmatrix} \begin{bmatrix} 0 \\ 2 \\ \end{bmatrix}$

(e) $\begin{bmatrix} 1 \\ 1 \\ 3 \end{bmatrix}, \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix}$

(a) Yes. linearly independent.
2pt

since $\begin{vmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{vmatrix} = 1 \neq 0$ 2pt/2

(c). $\begin{vmatrix} 2 & 3 & 2 \\ 1 & 2 & 2 \\ -2 & -2 & 0 \end{vmatrix}$
2pt

$= 2\cdot\begin{vmatrix} 2 & 2 \\ -2 & 0 \end{vmatrix} - 3\cdot\begin{vmatrix} 1 & 2 \\ -2 & 0 \end{vmatrix} + 2\cdot\begin{vmatrix} 1 & 2 \\ -2 & -2 \end{vmatrix}$

$= 2\cdot 4 - 3\cdot 4 + 2(-2+4)$

$= 8 - 12 + 4 = 0.$

No. linear dependent. 2pt/2

(e) $c_1 \cdot \begin{bmatrix} 1 \\ 1 \\ 3 \end{bmatrix} + c_2 \cdot \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ 1pt/3
3pt

Augmented matrix.

$\left[\begin{array}{cc|c} 1 & 0 & 0 \\ 1 & 2 & 0 \\ 3 & 1 & 0 \end{array}\right] \longrightarrow \left[\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 1 & 0 \end{array}\right]$

$$\rightarrow \left[\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

only has zero solution: $c_1 = c_2 = 0$. 1pt/3

Yes. linearly independent. 1pt/3.

8pt **8.** Determine whether the following vectors are linearly independent in P_3:

(a) $1, x^2, x^2 - 2$ **(b)** $2, x^2, x, 2x + 3$

(a). Suppose

4pt

$$c_1 \cdot 1 + c_2 \cdot x^2 + c_3 \cdot (x^2 - 2) = 0$$ 1pt/4

$$\Rightarrow c_1 - 2c_3 + (c_2 + c_3) \cdot x^2 = 0$$

$$\Rightarrow c_1 - 2c_3 = 0, \quad c_2 + c_3 = 0$$

There is a nonzero solution: 2pt/4

$$c_1 = 2, \quad c_2 = -1, \quad c_3 = 1$$

(or any other nonzero choice).

i.e.

$$2 \cdot 1 + (-1) \cdot x^2 + (x^2 - 2) = 0.$$

The vectors are linearly dependent 1pt/4.

(b). Suppose

4pt

4pt suppose

$$c_1 \cdot 2 + c_2 \cdot x^2 + c_3 x + c_4 \cdot (2x+3) = 0$$

1pt/4

$$\Rightarrow 2c_1 + 3c_4 + (c_3 + 2c_4) \cdot x + c_2 \cdot x^2 = 0$$

$$\Rightarrow 2c_1 + 3c_4 = 0, \quad c_3 + 2c_4 = 0, \quad c_2 = 0.$$

let $c_4 = 1$. $c_2 = 0$

$c_1 = -\frac{3}{2}$, $c_3 = -2$ is a nonzero solution

2pt/4 (or any other nonzero choice)

i.e.

$$-\frac{3}{2} \cdot 2 + 0 \cdot x^2 - 2 \cdot x + 1 \cdot (2x+3) = 0$$

The vectors are linearly dependent. 1pt/4.

Sec 3.4, 3, 4, 5, 14 (a)(b)(c)(d).

来自 <https://users.math.msu.edu/users/zhangshiwen/s19/homework.html>

6pt **3.** Consider the vectors

$$\mathbf{x}_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}, \quad \mathbf{x}_2 = \begin{bmatrix} 4 \\ 3 \end{bmatrix}, \quad \mathbf{x}_3 = \begin{bmatrix} 7 \\ -3 \end{bmatrix}$$

(a) Show that $\mathbf{x}_1$ and $\mathbf{x}_2$ form a basis for $\mathbb{R}^2$.

(b) Why must $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3$ be linearly dependent?

(c) What is the dimension of Span$(\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3)$?

(a). 2pt $X = (x_1, x_2) = \begin{bmatrix} 2 & 4 \\ 1 & 3 \end{bmatrix}$

$\det X = \begin{vmatrix} 2 & 4 \\ [illegible] & [illegible] \end{vmatrix} = 2 \cdot 3 - 4 \cdot 1 \neq 0$

2pt

$\det X = \begin{vmatrix} 2 & 4 \\ 1 & 3 \end{vmatrix} = 2\cdot 3 - 4\cdot 1 \neq 0$

Therefore, x_1, x_2 form a basis. 2pt/2

2pt (b) Yes. More than 2 vectors in $\mathbb{R}^2$ must be linearly dependent. 2pt/2

Or: $c_1 \cdot \begin{bmatrix} 2 \\ 1 \end{bmatrix} + c_2 \cdot \begin{bmatrix} 4 \\ 3 \end{bmatrix} + c_3 \begin{bmatrix} 7 \\ -3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ 1pt/2

$$\left[\begin{array}{ccc|c} 2 & 4 & 7 & 0 \\ 1 & 3 & -3 & 0 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 3 & -3 & 0 \\ 0 & -2 & 14 & 0 \end{array}\right]$$

$$\to \left[\begin{array}{ccc|c} 1 & 3 & -3 & 0 \\ 0 & 1 & -7 & 0 \end{array}\right]$$

The system has one free variable and therefore has nontrivial solution

Therefore, x_1, x_2, x_3 are linearly dependent. 1pt/2

2pt (c) $\mathrm{Span}(x_1, x_2, x_3) = \mathrm{Span}(x_1, x_2)$

x_1, x_2 form a basis for $\mathrm{Span}(x_1, x_2, x_3)$

Therefore, $\mathrm{Span}(x_1, x_2, x_3)$ has $\boxed{\text{dimension } 2}$. 2pt/2

Therefore, Span(X_1, X_2, X_3) has $\boxed{\text{dimension } 2}$. 2pt/2

4pt **4.** Given the vectors

$$\mathbf{x}_1 = \begin{bmatrix} 3 \\ -2 \\ 4 \end{bmatrix}, \quad \mathbf{x}_2 = \begin{bmatrix} -3 \\ 2 \\ -4 \end{bmatrix}, \quad \mathbf{x}_3 = \begin{bmatrix} -6 \\ 4 \\ -8 \end{bmatrix}$$

what is the dimension of Span($\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3$)?

It is easy to check that

$$X_2 = -X_1 \quad \text{and} \quad X_3 = -2X_1$$

1pt/4 1pt/4

Therefore,

$$\text{Span}(X_1, X_2, X_3) = \left\{ \alpha \cdot \begin{pmatrix} 3 \\ -2 \\ 4 \end{pmatrix} \Big/ \alpha \in \mathbb{R} \right\}$$

is spanned by X_1. 1pt/4

The $\boxed{\text{dimension is } 1}$. 1pt/4.

8pt **5.** Let

$$\mathbf{x}_1 = \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix}, \quad \mathbf{x}_2 = \begin{bmatrix} 3 \\ -1 \\ 4 \end{bmatrix}, \quad \mathbf{x}_3 = \begin{bmatrix} 2 \\ 6 \\ 4 \end{bmatrix}$$

(a) Show that $\mathbf{x}_1, \mathbf{x}_2$, and $\mathbf{x}_3$ are linearly dependent.

(b) Show that $\mathbf{x}_1$ and $\mathbf{x}_2$ are linearly independent.

(c) What is the dimension of Span($\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3$)?

(d) Give a geometric description of Span($\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3$).

2pt (a)

$$\det(X_1, X_2, X_3) = \begin{vmatrix} 2 & 3 & 2 \\ 1 & -1 & 6 \\ 3 & 4 & 4 \end{vmatrix}$$

$$\det(x_1, x_2, x_3) = \begin{vmatrix} 1 & -1 & 6 \\ 3 & 4 & 4 \end{vmatrix}$$

$$= 2 \cdot \begin{vmatrix} -1 & 6 \\ 4 & 4 \end{vmatrix} - 3 \cdot \begin{vmatrix} 1 & 6 \\ 3 & 4 \end{vmatrix} + 2 \cdot \begin{vmatrix} 1 & -1 \\ 3 & 4 \end{vmatrix}.$$

$$= 2 \cdot (-4 - 24) - 3 (4 - 18) + 2 (4 + 3)$$

$$= -56 + 42 + 14$$

$$= 0.$$

2pt/2

Therefore, x_1, x_2, x_3 are linearly independent

3pt (b).

$$c_1 \cdot \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix} + c_2 \cdot \begin{bmatrix} 3 \\ -1 \\ 4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

1pt/3

$$\Rightarrow \left[\begin{array}{cc|c} 2 & 3 & 0 \\ 1 & -1 & 0 \\ 3 & 4 & 0 \end{array}\right] \to \left[\begin{array}{cc|c} 1 & -1 & 0 \\ 0 & 5 & 0 \\ 0 & 5 & 0 \end{array}\right]$$

$$\to \left[\begin{array}{cc|c} 1 & -1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

$\Rightarrow c_1 = c_2 = 0$ 1pt/3

x_1, x_2 are linearly independent. 1pt/3.

2pt (c). $\mathrm{Span}(x_1, x_2, x_3)$ can be spanned only by x_1, x_2.

The dimension of $\mathrm{Span}(x_1, x_2, x_3)$ is 2.

2pt/2

1pt (d): x_1, x_2, x_3 are three vectors in $\mathbb{R}^3$ lying in the same plane spanned only by x_1, x_2. 1pt/1

(any reasonable explaination is ok).

15pt **14.** In each of the following, find the dimension of the subspace of P_3 spanned by the given vectors:

(a) $x, x-1, x^2+1$ 4

(b) $x, x-1, x^2+1, x^2-1$ 4

(c) $x^2, x^2-x-1, x+1$ 4 **(d)** $2x, x-2$ 3.

4pt (a): $c_1 \cdot x + c_2(x-1) + c_3(x^2+1) = 0$ 1pt/4

$\Rightarrow (-c_2 + c_3) + (c_1 + c_2) \cdot x + c_3 \cdot x^2 = 0$

$\Rightarrow -c_2 + c_3 = 0, \quad c_1 + c_2 = 0, \quad c_3 = 0$

$\Rightarrow c_1 = c_2 = c_3 = 0.$ 1pt/4

$x, x-1, x^2+1$ are linearly independent 1pt/4

$\mathrm{Span}(x, x-1, x^2+1)$ has $\boxed{\text{dimension } 3}$. 1pt/4

4pt (b) $c_1 x + c_2(x-1) + c_3(x^2+1) + c_4(x^2-1) = 0$

4pt (b) $c_1\cdot x + c_2(x-1) + c_3\cdot(x^2+1) + c_4\cdot(x^2-1) = 0$ 1pt/4

$\Rightarrow (-c_2+c_3-c_4) + (c_1+c_2)x + (c_3+c_4)\cdot x^2 = 0$

$$\Rightarrow \begin{aligned} -c_2 + c_3 - c_4 &= 0 \\ c_1 + c_2 &= 0 \\ c_3 + c_4 &= 0 \end{aligned}$$

$$\Rightarrow \begin{bmatrix} 0 & -1 & 1 & -1 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\rightarrow \left[\begin{array}{cccc|c} 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & -1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 \end{array}\right]$$

$$\rightarrow \left[\begin{array}{cccc|c} 1 & 0 & 0 & -2 & 0 \\ 0 & 1 & 0 & 2 & 0 \\ 0 & 0 & 1 & 1 & 0 \end{array}\right]$$

c_4 is a free variable for the system.

We can take $c_4 = 1$. then $\boxed{\begin{aligned} c_3 &= -1 \\ c_2 &= -2 \\ c_1 &= 2 \end{aligned}}$

is a nontrivial solution. 1pt/4 (or any nonzero choice)

The four vectors are linearly dependent 1pt/4

The four vectors are linearly dependent 1pt/4.

By Part (a), the first three are linearly independent.

Therefore,

Span$(x, x-1, x^2+1, x^2-1)$ has $\boxed{\text{dimension } 3.}$ 1pt/4.

4pt (c) It is easy to check that

$$x^2 = (x^2-x-1) + (x+1)$$

Therefore, the three vectors are linearly dependent 2pt/4.

And, x^2 and $x+1$ are linearly independent 1pt/4

since $c_1 x^2 + c_2(x+1) = 0$ (or any two of the three)

implies $c_1 = c_2 = 0$,

Therefore,

Span$(x^2, x^2-x-1, x+1)$ has $\boxed{\text{dimension } 2}$. 1pt/4.

3pt (d). $2x$ and $x-2$ are linearly independent 2pt/3.

since $c_1(2x) + c_2(x-2) = 0$

implies $c_1 = c_2 = 0$.

Span$(2x, x-2)$ has $\boxed{\text{dimension } 2}$. 1pt/3

$\mathrm{Span}(2X, X-2)$ has dimension 2. 1pt/3

Sec 3.5, 1 (a), 3 (b)

来自 <https://users.math.msu.edu/users/zhangshiwen/s19/homework.html>

1. For each of the following, find the transition matrix corresponding to the change of basis from $\{\mathbf{u}_1, \mathbf{u}_2\}$ to $\{\mathbf{e}_1, \mathbf{e}_2\}$.

(a) $\mathbf{u}_1 = (1, 1)^T, \quad \mathbf{u}_2 = (-1, 1)^T$

$$U = (u_1, u_2) = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$$ is the transition matrix from $\{u_1, u_2\}$ to $\{e_1, e_2\}$ 2pt/2

4pt **3.** Let $\mathbf{v}_1 = (3, 2)^T$ and $\mathbf{v}_2 = (4, 3)^T$. For each ordered basis $\{\mathbf{u}_1, \mathbf{u}_2\}$ given in Exercise 1, find the transition matrix from $\{\mathbf{v}_1, \mathbf{v}_2\}$ to $\{\mathbf{u}_1, \mathbf{u}_2\}$.

(b) $\mathbf{u}_1 = (1, 2)^T, \quad \mathbf{u}_2 = (2, 5)^T$

$$V = (v_1, v_2) = \begin{bmatrix} 3 & 4 \\ 2 & 3 \end{bmatrix}, \quad U = (u_1, u_2) = \begin{pmatrix} 1 & 2 \\ 2 & 5 \end{pmatrix}$$

$$U^{-1} = \frac{1}{5-4} \cdot \begin{bmatrix} 5 & -2 \\ -2 & 1 \end{bmatrix} = \begin{bmatrix} 5 & -2 \\ -2 & 1 \end{bmatrix}$$ 2pt/4.

The transition matrix from $\{v_1, v_2\}$ to $\{u_1, u_2\}$ is given by

$$U^{-1} \cdot V = \begin{bmatrix} 5 & -2 \\ 2 & 1 \end{bmatrix} \cdot \begin{bmatrix} 3 & 4 \\ 2 & 3 \end{bmatrix} = \begin{bmatrix} 11 & 14 \\ -4 & -5 \end{bmatrix}$$ 2pt/4

$u \cdot v = \begin{bmatrix} 2 & 1 \end{bmatrix} \cdot \begin{bmatrix} 2 & 3 \end{bmatrix} = \begin{bmatrix} -4 & -5 \end{bmatrix}$ 2pt/4.

(−2 pt if they write $v^{-1}(u)$).