HW6

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Sec 3.1, 3 (Check Axioms A1-A8), 10, 12, 13

来自 <https://users.math.msu.edu/users/zhangshiwen/s19/homework.html>

Definition

Let V be a set on which the operations of addition and scalar multiplication are defined. By this we mean that, with each pair of elements $\mathbf{x}$ and $\mathbf{y}$ in V, we can associate a unique element $\mathbf{x}+\mathbf{y}$ that is also in V, and with each element $\mathbf{x}$ in V and each scalar α, we can associate a unique element $\alpha\mathbf{x}$ in V. The set V together with the operations of addition and scalar multiplication is said to form a **vector space** if the following axioms are satisfied:

A1. $\mathbf{x}+\mathbf{y}=\mathbf{y}+\mathbf{x}$ for any $\mathbf{x}$ and $\mathbf{y}$ in V.
A2. $(\mathbf{x}+\mathbf{y})+\mathbf{z}=\mathbf{x}+(\mathbf{y}+\mathbf{z})$ for any $\mathbf{x}, \mathbf{y}$, and $\mathbf{z}$ in V.
A3. There exists an element $\mathbf{0}$ in V such that $\mathbf{x}+\mathbf{0}=\mathbf{x}$ for each $\mathbf{x} \in V$.
A4. For each $\mathbf{x} \in V$, there exists an element $-\mathbf{x}$ in V such that $\mathbf{x}+(-\mathbf{x})=\mathbf{0}$.
A5. $\alpha(\mathbf{x}+\mathbf{y})=\alpha\mathbf{x}+\alpha\mathbf{y}$ for each scalar α and any $\mathbf{x}$ and $\mathbf{y}$ in V.
A6. $(\alpha+\beta)\mathbf{x}=\alpha\mathbf{x}+\beta\mathbf{x}$ for any scalars α and β and any $\mathbf{x} \in V$.
A7. $(\alpha\beta)\mathbf{x}=\alpha(\beta\mathbf{x})$ for any scalars α and β and any $\mathbf{x} \in V$.
A8. $1\mathbf{x}=\mathbf{x}$ for all $\mathbf{x} \in V$.

8pt **3.** Let C be the set of complex numbers. Define addition on C by

$$(a+bi)+(c+di)=(a+c)+(b+d)i$$

and define scalar multiplication by

$$\alpha(a+bi)=\alpha a+\alpha bi$$

for all real numbers α. Show that C is a vector space with these operations.

1pt each axiam.

A1. $(a+bi)+(c+di)=(a+c)+(b+d)i$

$=(c+di)+(a+bi)$. 1pt

A2. $\big((a+bi)+(c+di)\big)+(e+fi)$

$=\big((a+c)+(b+d)i\big)+(e+fi)$

$=(a+c+e)+(b+d+f)i$ 1pt

$(a+bi)+\big((c+di)+(e+fi)\big)$

$= (a+bi) + \left((c+e) + (d+f)i \right)$

$= (a+c+e) + (b+d+f)i$

A3. zero vector $\bar{0} = 0 + 0i$

$(a+bi) + (0+0i) = (a+0) + (b+0)i$

$= a + bi.$ 1pt

A4. For $\bar{x} = a+bi$, $-\bar{x} = (-a) + (-b)i$

since

$a+bi + ((-a)+(-b)i) = (a-a) + (b-b)i$

$= 0 + 0 \cdot i$

$= \bar{0}.$ 1pt

A5. $\alpha \left((a+bi) + (c+di) \right)$

$= \alpha \cdot \left((a+c) + (b+d)i \right)$

$= \alpha \cdot (a+c) + \alpha (b+d) i$

$= (\alpha a + \alpha c) + (\alpha b + \alpha \cdot d) i$

$= (\alpha a + \alpha b i) + (\alpha \cdot c + \alpha d i)$

$= \alpha \cdot (a+bi) + \alpha \cdot (c+di).$ 1pt

A6. $(\alpha+\beta) \cdot (a+bi)$

$(\alpha+\beta)\cdot(a+bi)$

$$= (\alpha+\beta)a + (\alpha+\beta)bi$$

$$= \underline{\alpha \cdot a} + \beta \cdot a + (\underline{\alpha \cdot b} + \beta \cdot b) \cdot i$$

$$= \alpha a + \alpha \cdot bi + (\beta \cdot a + \beta bi)$$

$$= \alpha(a+bi) + \beta \cdot (a+bi).$$

1pt

A7. $(\alpha\beta)(a+bi)$

$$= (\alpha\beta) \cdot a + (\alpha\beta) bi$$

$$= \alpha \cdot (\beta a) + \alpha \cdot (\beta b i).$$

$$= \alpha \cdot (\beta a + \beta bi)$$

$$= \alpha \cdot (\beta \cdot (a+bi))$$

1pt

A8. $1 \cdot (a+bi) = 1 \cdot a + 1 \cdot bi$

$$= a + bi$$

1pt

10. Let S be the set of all ordered pairs of real numbers. Define scalar multiplication and addition on S by

4pt

$$\alpha(x_1, x_2) = (\alpha x_1, \alpha x_2)$$

$$(x_1, x_2) \oplus (y_1, y_2) = (x_1 + y_1, 0)$$

We use the symbol $\oplus$ to denote the addition operation for this system in order to avoid confusion with the usual addition $\mathbf{x} + \mathbf{y}$ of row vectors. Show that S, together with the ordinary scalar multiplication and the addition operation $\oplus$, is not a vector space. Which of the eight axioms fail to hold?

A3. There is no zero vector in this space.

A5. There is no zero vector in this space.

Take $\bar{x} = (0,1)$, then for any $\bar{y} = (y_1, y_2)$

$\bar{x} \oplus \bar{y} = (0,1) \oplus (y_1, y_2) = (y_1, 0)$ (-1pt if they assume the zero vector is $(0,0)$.)

which can not be equal to $(0,1)$.

Therefore, all $\bar{y}$ doesnot satisfy axiom 3. 4pt.

12. Let R^+ denote the set of positive real numbers. Define the operation of scalar multiplication, denoted $\circ$, by 10pt

$$\alpha \circ x = x^{\alpha}$$

for each $x \in R^+$ and for any real number α. Define the operation of addition, denoted $\oplus$, by

$$x \oplus y = x \cdot y \qquad \text{for all} \qquad x, y \in R^+$$

Thus, for this system, the scalar product of -3 times $\frac{1}{2}$ is given by

$$-3 \circ \frac{1}{2} = \left(\frac{1}{2}\right)^{-3} = 8$$

and the sum of 2 and 5 is given by

$$2 \oplus 5 = 2 \cdot 5 = 10$$

Is R^+ a vector space with these operations? Prove your answer.

Yes. R^+ is closed under $\circ$ and $\oplus$

since $\alpha \circ x = x^{\alpha} \in \mathbb{R}^+$ 1pt (closed under operations)

$x \oplus y = x \cdot y \in \mathbb{R}^+$ 1pt

for all $x, y \in \mathbb{R}^+$.

A1. $x \oplus y = xy = yx = y \oplus x.$ 1pt

A2. $(x \oplus y) \oplus z = (xy) \oplus z$

$= (xy) \cdot z$

$= x \cdot (yz)$ 1pt

$= x \oplus (y \oplus z).$

A3. $x \oplus 1 = x \cdot 1 = x$ 1pt

Therefore, the scalar 1 is the zero vector in $\mathbb{R}^+$.

A4. For any $x \in \mathbb{R}^+$,

$x \oplus (x^{-1}) = x \cdot x^{-1} = 1$ (zero vector)

Therefore, x^{-1} is the additive inverse of x. 1pt

A5. $\alpha \circ (x \oplus y) = \alpha \circ (xy)$

$= (xy)^{\alpha} = x^{\alpha} \cdot y^{\alpha}$

$(\alpha \circ x) \oplus (\alpha y) = x^{\alpha} \oplus y^{\alpha} = x^{\alpha} \cdot y^{\alpha}$ 1pt

Therefore, $\alpha \circ (x \oplus y) = (\alpha \circ x) \oplus (\alpha \circ y)$.

A6. $(\alpha + \beta) \circ x = x^{\alpha + \beta}$

1pt

A6. $(\alpha+\beta)\circ x = x^{\alpha+\beta}$

$$= x^{\alpha} \cdot x^{\beta}$$ 1pt

$$= (\alpha\circ x) \oplus (\beta\circ x).$$

A7. $(\alpha\beta)\circ x = x^{\alpha\beta}$

$$\alpha\circ(\beta\circ x) = \alpha\circ(x^{\beta})$$

$$= (x^{\beta})^{\alpha} = x^{\alpha\beta}$$

$$= (\alpha\beta)\circ x.$$ 1pt

A8. $1\circ x = x^{1} = x.$ 1pt

13. Let R denote the set of real numbers. Define scalar multiplication by

4pt

$$\alpha x = \alpha \cdot x \quad \text{(the usual multiplication of real numbers)}$$

and define addition, denoted $\oplus$, by

$$x \oplus y = \max(x, y) \quad \text{(the maximum of the two numbers)}$$

Is R a vector space with these operations? Prove your answer.

No. The space does not have a zero vector.

Suppose not

Assume y is the zero vector.

For $x < y$,

For $x < y$,

$x \oplus y = \max(x, y) = y \neq x$. contradiction. 4pt

(Full credits if they assume $y = 0$ and start with $x < 0$).

- **Sec 3.2, 1 (a)(e), 2 (a), 3 (a)(f), 4 (b)(c), 8, 11 (a)(c), 13 (a).**

6pt **1.** Determine whether the following sets form subspaces of $\mathbb{R}^2$:

(a) $\{(x_1, x_2)^T \mid x_1 + x_2 = 0\}$
(b) $\{(x_1, x_2)^T \mid x_1 x_2 = 0\}$
(c) $\{(x_1, x_2)^T \mid x_1 = 3x_2\}$
(d) $\{(x_1, x_2)^T \mid |x_1| = |x_2|\}$
(e) $\{(x_1, x_2)^T \mid x_1^2 = x_2^2\}$

(a). Yes. (conclusion 1pt)

3pt For $\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$ such that

$x_1 + x_2 = 0$ and $y_1 + y_2 = 0$,

$\alpha \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} \alpha x_1 \\ \alpha x_2 \end{pmatrix}$ satisfies $\alpha x_1 + \alpha x_2 = 0$ 1pt

$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} x_1 + y_1 \\ x_2 + y_2 \end{pmatrix}$ satisfies $(x_1 + y_1) + (x_2 + y_2) = 0$ 1pt.

3pt (e). No. Conclusion 1pt $\begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \end{pmatrix}$ are in the set $\left\{ \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \middle| x_1^2 = x_2^2 \right\}$.

3pt Conclusion 1pt

$\begin{pmatrix}1\\1\end{pmatrix}+\begin{pmatrix}1\\-1\end{pmatrix}=\begin{pmatrix}2\\0\end{pmatrix}$. $2^2 \neq 0^2$ their sum

is not in the set. (any counterexample 2pt).

3pt **2.** Determine whether the following sets form subspaces of $\mathbb{R}^3$:

(a) $\{(x_1, x_2, x_3)^T \mid x_1 + x_3 = 1\}$

No. $\begin{pmatrix}1\\0\\0\end{pmatrix}$ satisfies $1+0=1$. work. 2pt.

Conclusion 1pt

$2\begin{pmatrix}1\\0\\0\end{pmatrix}=\begin{pmatrix}2\\0\\0\end{pmatrix}$ does not since $2+0=2\neq 1$.

6pt **3.** Determine whether the following are subspaces of $\mathbb{R}^{2\times 2}$:

(a) The set of all 2×2 diagonal matrices

(b) The set of all 2×2 triangular matrices

(c) The set of all 2×2 lower triangular matrices

(d) The set of all 2×2 matrices A such that $a_{12} = 1$

(e) The set of all 2×2 matrices B such that $b_{11} = 0$

(f) The set of all symmetric 2×2 matrices

(a). Yes. $A=\begin{pmatrix}a_{11} & 0\\ 0 & a_{22}\end{pmatrix}$, $B=\begin{pmatrix}b_{11} & 0\\ 0 & b_{22}\end{pmatrix}$.
3pt 1pt

3pt 1pt $\ldots \; 0 \quad a_{22}) , \ldots \; 0 \quad b_{22})$.

$$\alpha \cdot A = \begin{pmatrix} \alpha \cdot a_{11} & 0 \\ 0 & \alpha \cdot a_{22} \end{pmatrix}$$ is still diagonal. 1pt

$$A + B = \begin{pmatrix} a_{11}+b_{11} & 0 \\ 0 & a_{22}+b_{22} \end{pmatrix}$$ is still diagonal 1pt

3pt (c).

Yes. 1pt

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}, \quad B = \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix}$$

$$a_{12} = a_{21} \qquad b_{12} = b_{21}$$

$$\alpha \cdot A = \begin{pmatrix} \alpha \cdot a_{11} & \alpha \cdot a_{12} \\ \alpha \cdot a_{21} & \alpha \cdot a_{22} \end{pmatrix}$$ is symmetric since 1pt

$$\alpha \cdot a_{12} = \alpha \cdot a_{21}$$

$$A + B = \begin{pmatrix} a_{11}+b_{11} & a_{12}+b_{12} \\ a_{21}+b_{21} & a_{22}+b_{22} \end{pmatrix}$$ is symmetric since

$$a_{12} + b_{12} = a_{21} + b_{21}.$$ 1pt

10pt **4.** Determine the null space of each of the following matrices: (b) (c)

(a) $\begin{bmatrix} 2 & 1 \\ 3 & 2 \end{bmatrix}$

(b) $\begin{bmatrix} 1 & 2 & -3 & -1 \\ -2 & -4 & 6 & 3 \end{bmatrix}$

5pt

(b).

$$\left[\begin{array}{cccc|c} 1 & 2 & -3 & -1 & 0 \\ -2 & -4 & 6 & 3 & 0 \end{array}\right]$$

correct augmented matrix 1pt/5

Row② + 2·Row①

$$\longrightarrow \left[\begin{array}{cccc|c} 1 & 2 & -3 & -1 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{array}\right]$$

$$\longrightarrow \left[\begin{array}{cccc|c} 1 & 2 & -3 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{array}\right]$$

$$\begin{cases} x_1 + 2x_2 - 3x_3 = 0 \\ x_4 = 0 \end{cases}$$

$$\Rightarrow \begin{cases} x_1 = -2x_2 + 3x_3 \\ x_4 = 0 \end{cases}$$

correct solution 2pt/5

let $x_2 = \alpha$, $x_3 = \beta$

$\begin{pmatrix} x_1 \\ \end{pmatrix}$ $\begin{pmatrix} -2\alpha + 3\beta \\ \end{pmatrix}$ $\begin{pmatrix} -2 \\ \end{pmatrix}$ $\begin{pmatrix} 3 \\ \end{pmatrix}$

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} -2\alpha+3\beta \\ \alpha \\ \beta \\ 0 \end{pmatrix} = \alpha \cdot \begin{pmatrix} -2 \\ 1 \\ 0 \\ 0 \end{pmatrix} + \beta \cdot \begin{pmatrix} 3 \\ 0 \\ 1 \\ 0 \end{pmatrix}$$

$$N(A) = \left\{ \underbrace{\alpha \cdot \begin{pmatrix} -2 \\ 1 \\ 0 \\ 0 \end{pmatrix} + \beta \cdot \begin{pmatrix} 3 \\ 0 \\ 1 \\ 0 \end{pmatrix}}_{} \,\middle|\, \alpha, \beta \in \mathbb{R} \right\}.$$

2pt/5.

5pt **(c)** $\begin{bmatrix} 1 & 3 & -4 \\ 2 & -1 & -1 \\ -1 & -3 & 4 \end{bmatrix}$

$$\left[\begin{array}{ccc|c} 1 & 3 & -4 & 0 \\ 2 & -1 & -1 & 0 \\ -1 & -3 & 4 & 0 \end{array}\right]$$

1pt/5

$$\longrightarrow \left[\begin{array}{ccc|c} 1 & 3 & -4 & 0 \\ 0 & -7 & 7 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$\longrightarrow \left[\begin{array}{ccc|c} 1 & 3 & -4 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$\longrightarrow \left[\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$\begin{cases} x_1 - x_3 = 0 \\ x_2 - x_3 = 0 \end{cases}$$. Let $x_3 = \alpha$.

2pt/5

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} \alpha \\ \alpha \\ \alpha \end{pmatrix} = \alpha \cdot \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

$$N(A) = \left\{ \alpha \cdot \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \;\middle|\; \alpha \in \mathbb{R} \right\}$$

2pt/5.

4pt **8.** Let A be a fixed vector in $\mathbb{R}^{n\times n}$ and let S be the set of all matrices that commute with A, that is,

$$S = \{B \mid AB = BA\}$$

Show that S is a subspace of $\mathbb{R}^{n\times n}$.

A is fixed.

For any $B, C \in S$.

$AB = BA$, and $AC = C \cdot A$

For any scalar α,

$$A(\alpha B) = \alpha(AB) = \alpha \cdot (BA) = (\alpha B) \cdot A$$

ie., $\alpha \cdot B$ also commutes with A

$\Rightarrow \alpha \cdot B \in S$. 2pt.

$$A\cdot(B+C) = AB + AC = BA + CA$$
$$= (B+C)A.$$

ie, $B+C$ also commutes with A

$\Rightarrow B+C \in S$. 2pt

Therefore, S is a subspace.

7pt **11.** Determine whether the following are spanning sets for $\mathbb{R}^2$:

(a) $\left\{ \begin{pmatrix} 2 \\ 1 \end{pmatrix}, \begin{pmatrix} 3 \\ 2 \end{pmatrix} \right\}$ **(b)** $\left\{ \begin{pmatrix} 2 \\ 3 \end{pmatrix}, \begin{pmatrix} 4 \\ 6 \end{pmatrix} \right\}$

(c) $\left\{ \begin{pmatrix} -2 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 3 \end{pmatrix}, \begin{pmatrix} 2 \\ 4 \end{pmatrix} \right\}$

(a) Yes. 1pt

3pt

For any vector $\begin{pmatrix} b_1 \\ b_2 \end{pmatrix} \in \mathbb{R}^2$, consider

$$x_1 \cdot \begin{pmatrix} 2 \\ 1 \end{pmatrix} + x_2 \begin{pmatrix} 3 \\ 2 \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}$$

$$\Leftrightarrow \begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} \quad (*)$$

$A = \begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix}$ (the coefficient matrix)

is invertible since $\det A = 2\cdot 2 - 3\cdot 1 = 1 \neq 0$ 2pt

Therefore, equation $(*)$ always has a solution

for any $\begin{pmatrix} b_1 \\ b_2 \end{pmatrix}$.

therefore, the two vectors span $\mathbb{R}^2$.

(c). For any $\begin{pmatrix} b_1 \\ b_2 \end{pmatrix} \in \mathbb{R}^2$, consider 4pt

$$x_1 \cdot \begin{bmatrix} -2 \\ 1 \end{bmatrix} + x_2 \cdot \begin{bmatrix} 1 \\ 3 \end{bmatrix} + x_3 \cdot \begin{bmatrix} 2 \\ 4 \end{bmatrix} = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}$$ 1pt/4 for equation.

$$\Leftrightarrow \left[\begin{array}{ccc|c} -2 & 1 & 2 & b_1 \\ 1 & 3 & 4 & b_2 \end{array}\right]$$ (or matrix).

$$\xrightarrow{\text{interchange ① and ②}} \left[\begin{array}{ccc|c} 1 & 3 & 4 & b_2 \\ -2 & 1 & 2 & b_1 \end{array}\right]$$

$$\xrightarrow{\text{Row ② + 2 Row ①}} \left[\begin{array}{ccc|c} 1 & 3 & 4 & b_2 \\ 0 & 7 & 10 & 2b_2 + b_1 \end{array}\right]$$ 2pt/4 for reduction (not necessary

$\left(0 \quad 7 \quad 10 \;\vdots\; 2b_2+b_1\right]$ 2pt/4 for reduction (not neccessary to be row echelon)

The system has two lead variables x_1, x_2 and one free variable x_3 for any b_1, b_2

The system is consistent (1pt/4.)

Therefore, the three vectors span $\mathbb{R}^2$

13. Given

$$\mathbf{x}_1 = \begin{bmatrix} -1 \\ 2 \\ 3 \end{bmatrix}, \quad \mathbf{x}_2 = \begin{bmatrix} 3 \\ 4 \\ 2 \end{bmatrix},$$

$$\mathbf{x} = \begin{bmatrix} 2 \\ 6 \\ 6 \end{bmatrix}, \quad \mathbf{y} = \begin{bmatrix} -9 \\ -2 \\ 5 \end{bmatrix}$$

(a) Is $\mathbf{x} \in \text{Span}(\mathbf{x}_1, \mathbf{x}_2)$?

4pt ✓

$\text{Span}(\overline{x}_1, \overline{x}_2) = \{ c_1 \cdot \overline{x}_1 + c_2 \cdot \overline{x}_2 \mid c_1, c_2 \in \mathbb{R} \}$

It is equivalent to ask: are there c_1, c_2 such that

$$c_1 \cdot \begin{bmatrix} -1 \\ 2 \\ 3 \end{bmatrix} + c_2 \cdot \begin{bmatrix} 3 \\ 4 \\ 2 \end{bmatrix} = \begin{bmatrix} 2 \\ 6 \\ 6 \end{bmatrix}$$ 1p/4 for setting up.

$$\Leftrightarrow \begin{bmatrix} -1 & 3 \\ 2 & 4 \\ 3 & 2 \end{bmatrix} \cdot \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 2 \\ 6 \\ 6 \end{bmatrix}$$

$$\left[\begin{array}{cc|c} -1 & 3 & 2 \\ 2 & 4 & 6 \\ 3 & 2 & 6 \end{array}\right]$$

$$\longrightarrow \left[\begin{array}{cc|c} -1 & 3 & 2 \\ 0 & 10 & 10 \\ 0 & 11 & 12 \end{array}\right]$$

$$\longrightarrow \left[\begin{array}{cc|c} -1 & 3 & 2 \\ 0 & 1 & 1 \\ 0 & 11 & 12 \end{array}\right]$$

$$\longrightarrow \left[\begin{array}{cc|c} -1 & 3 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{array}\right]$$ 2p/4 for reduction.

The system is inconsistent (no solution) 1pt/4

$\bar{x}$ is not in the span of $\bar{x}_1, \bar{x}_2$.