HW5

2019年2月12日 21:12

Sec 2.1, 1(a)(b), 2(a)(b), 3(a)(c)(e)(g), 5, 6;

1. Let

$$A = \begin{bmatrix} 3 & 2 & 4 \\ 1 & -2 & 3 \\ 2 & 3 & 2 \end{bmatrix}$$

9pt **(a)** Find the values of $\det(M_{21})$, $\det(M_{22})$, and $\det(M_{23})$.

(b) Find the values of A_{21}, A_{22}, and A_{23}.

(a) 6pt

$$M_{21} = \begin{bmatrix} 3 & 2 & 4 \\ 1 & -2 & 3 \\ 2 & 3 & 2 \end{bmatrix} = \begin{bmatrix} 2 & 4 \\ 3 & 2 \end{bmatrix}$$

$$\det M_{21} = \begin{vmatrix} 2 & 4 \\ 3 & 2 \end{vmatrix} = 2\cdot 2 - 4\cdot 3 = -8$$ 2pt/6

$$M_{22} = \begin{bmatrix} 3 & 2 & 4 \\ 1 & -2 & 3 \\ 2 & 3 & 2 \end{bmatrix} = \begin{bmatrix} 3 & 4 \\ 2 & 2 \end{bmatrix}$$

$$\det M_{22} = \begin{vmatrix} 3 & 4 \\ 2 & 2 \end{vmatrix} = 3\cdot 2 - 4\cdot 2 = -2$$ 2pt/6

$$M_{23} = \begin{bmatrix} 3 & 2 & 4 \\ 1 & -2 & 3 \\ 2 & 3 & 2 \end{bmatrix} = \begin{bmatrix} 3 & 2 \\ 2 & 3 \end{bmatrix}$$

$$\det M_{23} = \begin{vmatrix} 3 & 2 \\ 2 & 3 \end{vmatrix} = 3\cdot 3 - 2\cdot 2 = 5$$ 2pt/6

(b). $A_{21} = (-1)^{2+1} \det M_{21} = -(-8) = 8$ 1pt/3

3pt

$A_{22} = (-1)^{2+2} \det M_{22} = -2$ 1pt/3

$A_{23} = (-1)^{2+3} \det M_{23} = -5$ 1pt/3

4pt **2.** Use determinants to determine whether the following 2×2 matrices are nonsingular:

(a) $\begin{bmatrix} 3 & 5 \\ 2 & 4 \end{bmatrix}$ **(b)** $\begin{bmatrix} 3 & 6 \\ 2 & 4 \end{bmatrix}$

(a) $\begin{vmatrix} 3 & 5 \\ 2 & 4 \end{vmatrix} = 3 \cdot 4 - 2 \cdot 5 = 2 \neq 0$ 1pt/2

2pt

nonsingular 1pt/2

(b) $\begin{vmatrix} 3 & 6 \\ 2 & 4 \end{vmatrix} = 3 \cdot 4 - 2 \cdot 6 = 0$ 1pt/2

2pt

singular 1pt/2

3. Evaluate the following determinants:

10pt (a) (c), (e) (g)

(a) $\begin{vmatrix} 3 & 5 \\ -2 & -3 \end{vmatrix}$

1pt (a) $\begin{vmatrix} 3 & 5 \\ -2 & -3 \end{vmatrix} = 3(-3) - 5 \cdot (-2)$

$= -9 + 10 = 1$

2pt **(c)** $\begin{vmatrix} 3 & 1 & 2 \\ 2 & 4 & 5 \\ 2 & 4 & 5 \end{vmatrix}$

$$\begin{vmatrix} 3 & 1 & 2 \\ 2 & 4 & 5 \\ 2 & 4 & 5 \end{vmatrix}$$

$$= 3 \cdot \begin{vmatrix} 4 & 5 \\ 4 & 5 \end{vmatrix} - \begin{vmatrix} 2 & 5 \\ 2 & 5 \end{vmatrix} + 2 \begin{vmatrix} 2 & 4 \\ 2 & 4 \end{vmatrix}$$

$$= 3(4 \cdot 5 - 4 \cdot 5) - (2 \cdot 5 - 2 \cdot 5) + 2 \cdot (2 \cdot 4 - 2 \cdot 4)$$

$$= 0$$ 2pt

If one writes directly $\det \begin{vmatrix} \vdots & \vdots \end{vmatrix} = 0$, then one needs the statement "the second and the third raw are the same".

Otherwise, only 1pt/2.

(e) $\begin{vmatrix} 1 & 3 & 2 \\ 4 & 1 & -2 \\ 2 & 1 & 3 \end{vmatrix}$ 3pt

$$= 1 \cdot \begin{vmatrix} 1 & -2 \\ 1 & 3 \end{vmatrix} - 3 \cdot \begin{vmatrix} 4 & -2 \\ 2 & 3 \end{vmatrix} + 2 \begin{vmatrix} 4 & 1 \\ 2 & 1 \end{vmatrix}$$

$$= 3-(-2) - 3(12-(-2)\cdot 2) + 2\cdot(4-2)$$

$$= 5 - 3(16) + 2\cdot 2$$

$$= 5 - 48 + 4$$

$$= -39 \qquad 3pt.$$

(full credit as long as they have the correct determinant with work.

1pt if without any work.

(g) 4pt
$$\begin{vmatrix} 2 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 1 & 6 & 2 & 0 \\ 1 & 1 & -2 & 3 \end{vmatrix}$$

$$= 2\cdot\begin{vmatrix} 1 & 0 & 0 \\ 6 & 2 & 0 \\ 1 & -2 & 3 \end{vmatrix} - \begin{vmatrix} 0 & 1 & 0 \\ 1 & 6 & 2 \\ 1 & 1 & -2 \end{vmatrix}$$

$$= 2\cdot 6 - 4 = 8$$

answer 2pt/4

work 2pt/4.

$$\begin{vmatrix} 1 & 0 & 0 \\ 6 & 2 & 0 \\ 1 & -2 & 3 \end{vmatrix} = 1\cdot 2\cdot 3 = 6 \text{ (triangular form).}$$

work 2pt/4.

$$\begin{vmatrix} 1 & 0 & 0 \\ 6 & 2 & 0 \\ 1 & -2 & 3 \end{vmatrix} = 1\cdot 2\cdot 3 = 6 \quad \text{(triangular form).}$$

$$\begin{vmatrix} 0 & 1 & 0 \\ 1 & 6 & 2 \\ 1 & 1 & -2 \end{vmatrix} = -\begin{vmatrix} 1 & 2 \\ 1 & -2 \end{vmatrix} = -(-2-2) = 4.$$

5. Evaluate the following determinant. Write your answer as a polynomial in x:

4pt.

$$\begin{vmatrix} a-x & b & c \\ 1 & -x & 0 \\ 0 & 1 & -x \end{vmatrix}$$

$$\begin{vmatrix} a-x & b & c \\ 1 & -x & 0 \\ 0 & 1 & -x \end{vmatrix} \quad \text{(expand w.r.t. the first column)}$$

$$= (a-x)\cdot\begin{vmatrix} -x & 0 \\ 1 & -x \end{vmatrix} - \begin{vmatrix} b & c \\ 1 & -x \end{vmatrix} + 0$$

$$= (a-x)\cdot(-x)\cdot(-x) - \big(b(-x) - c\big)$$

$$= (a-x)\cdot x^2 - (-bx - c) \quad \text{(lose 1pt if ends here)}$$

$$= -x^3 + ax^2 + bx + c$$

$$= \underbrace{-x^3}_{1pt/4} + \underbrace{ax^2}_{1pt/4} + \underbrace{bx}_{1pt/4} + \underbrace{c}_{1pt/4}$$

6. Find all values of λ for which the following determinant will equal 0:

4 pt

$$\begin{vmatrix} 2-\lambda & 4 \\ 3 & 3-\lambda \end{vmatrix}$$

$$\begin{vmatrix} 2-\lambda & 4 \\ 3 & 3-\lambda \end{vmatrix} = (2-\lambda)(3-\lambda)-12$$

$$= \lambda^2 - 5\lambda + 6 - 12$$

$$= \lambda^2 - 5\lambda - 6 \quad 2pt/4$$

$$\lambda^2 - 5\lambda - 6 = 0$$

$$\Leftrightarrow (\lambda-6)(\lambda+1) = 0$$

$$\lambda = 6, \quad \lambda = -1.$$

1pt/4 1pt/4.

(lose one pt if $\lambda=-6, \lambda=1$)

Sec 2.2, 1 (a)(c); 3 (c)(e)(f); 5, 6, 7

4 pt **1.** Evaluate each of the following determinants by inspection.

(a) $\begin{vmatrix} 0 & 0 & 3 \\ 0 & 4 & 1 \\ 2 & 3 & 1 \end{vmatrix}$

2pt

$$\begin{vmatrix} 0 & 0 & 3 \\ 0 & 4 & 1 \\ 2 & 3 & 1 \end{vmatrix} = 3\begin{vmatrix} 0 & 4 \\ 2 & 3 \end{vmatrix}$$

$$= 3\cdot(0-2\cdot4) = -24$$

or

$$\begin{vmatrix} 0 & 0 & 3 \\ 0 & 4 & 1 \\ 2 & 3 & 1 \end{vmatrix} = 2\cdot\begin{vmatrix} 0 & 3 \\ 4 & 1 \end{vmatrix}$$ 2pt

$$= 2(-3\cdot4) = -24$$

(full credit if one writes $\det|\cdot\cdot| = -2\cdot4\cdot3 = -24$ directly)

(c) $\begin{vmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{vmatrix}$

2pt

$$\begin{vmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{vmatrix}$$ (Expand w.r.t the first row)

$$\begin{vmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{vmatrix}$$ (Expand w.r.t the first row)

$$= -\begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} = -1$$ 2pt

(no work needed)

10pt

3. For each of the following, compute the determinant and state whether the matrix is singular or nonsingular:

(a) $\begin{bmatrix} 3 & 1 \\ 6 & 2 \end{bmatrix}$ **(b)** $\begin{bmatrix} 3 & 1 \\ 4 & 2 \end{bmatrix}$

(c) $\begin{bmatrix} 3 & 3 & 1 \\ 0 & 1 & 2 \\ 0 & 2 & 3 \end{bmatrix}$ **(d)** $\begin{bmatrix} 2 & 1 & 1 \\ 4 & 3 & 5 \\ 2 & 1 & 2 \end{bmatrix}$

(e) $\begin{bmatrix} 2 & -1 & 3 \\ -1 & 2 & -2 \\ 1 & 4 & 0 \end{bmatrix}$

(f) $\begin{bmatrix} 1 & 1 & 1 & 1 \\ 2 & -1 & 3 & 2 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 7 & 3 \end{bmatrix}$

3pt (c).

$$\begin{vmatrix} 3 & 3 & 1 \\ 0 & 1 & 2 \\ 0 & 2 & 3 \end{vmatrix} = 3 \cdot \begin{vmatrix} 1 & 2 \\ 2 & 3 \end{vmatrix}$$

$$= 3 \cdot (3-4) = -3$$ 2pt/3

nonsingular 1pt/3

nonsingular 1pt/3

3pt (e)

$$\begin{vmatrix} 2 & -1 & 3 \\ -1 & 2 & -2 \\ 1 & 4 & 0 \end{vmatrix}$$

$$= 2 \cdot \begin{vmatrix} 2 & -2 \\ 4 & 0 \end{vmatrix} - (-1) \cdot \begin{vmatrix} -1 & -2 \\ 1 & 0 \end{vmatrix} + 3 \cdot \begin{vmatrix} -1 & 2 \\ 1 & 4 \end{vmatrix}$$

$$= 2(0+8) + (0+2) + 3 \cdot (-4-2)$$

$$= 16 + 2 - 18$$

$$= 0$$

2pt/3

singular 1pt/3

(f) 4pt

$$\begin{vmatrix} 1 & 1 & 1 & 1 \\ 2 & -1 & 3 & 2 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 7 & 3 \end{vmatrix}$$

(1st column)

$$= 1 \cdot \begin{vmatrix} -1 & 3 & 2 \\ 1 & 2 & 1 \\ 0 & 7 & 3 \end{vmatrix} - 2 \cdot \begin{vmatrix} 1 & 1 & 1 \\ 1 & 2 & 1 \\ 0 & 7 & 3 \end{vmatrix}$$

$$= 6 - 2 \cdot 3 = 0$$ 3pt/4 . singular 1pt/4

$$\begin{vmatrix} -1 & 3 & 2 \\ 1 & 2 & 1 \\ 0 & 7 & 3 \end{vmatrix} = -\begin{vmatrix} 2 & 1 \\ 7 & 3 \end{vmatrix} - \begin{vmatrix} 3 & 2 \\ 7 & 3 \end{vmatrix}$$

$$= -(6-7) - (9-14)$$

$$= 1 + 5 = 6$$

$$\begin{vmatrix} 1 & 1 & 1 \\ 1 & 2 & 1 \\ 0 & 7 & 3 \end{vmatrix} = \begin{vmatrix} 2 & 1 \\ 7 & 3 \end{vmatrix} - \begin{vmatrix} 1 & 1 \\ 7 & 3 \end{vmatrix}$$

$$= 6-7 - (3-7)$$

$$= -1 + 4 = 3$$

5. Let A be an $n \times n$ matrix and α a scalar. Show that

5pt

$$\det(\alpha A) = \alpha^n \det(A)$$

Proof 1:

$$A = (a_{ij}) = \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \vdots \\ a_{n1} & \cdots & a_{nn} \end{bmatrix}$$

$$\alpha A = (\alpha a_{ij}) = \begin{bmatrix} \alpha \cdot a_{11} & \cdots & \alpha \cdot a_{1n} \\ \alpha a_{21} & \cdots & \alpha \cdot a_{2n} \\ \cdots & \cdots & \cdots \\ \alpha \cdot a_{n1} & \cdots & \alpha \cdot a_{nn} \end{bmatrix}$$

To compute $\det(\alpha A)$, by the property of the elementary matrix, typ II:

$$\begin{vmatrix} \alpha \cdot a_{11} & \cdots & \alpha \cdot a_{1n} \\ \alpha \cdot a_{21} & \cdots & \alpha \cdot a_{2n} \\ \cdots & \cdots & \cdots \\ \alpha \cdot a_{n1} & \cdots & \alpha \cdot a_{nn} \end{vmatrix} = \alpha \cdot \begin{vmatrix} a_{11} & \cdots & a_{1n} \\ \alpha \cdot a_{21} & \cdots & \alpha a_{2n} \\ \vdots & & \\ \alpha a_{n1} & \cdots & \alpha \cdot a_{nn} \end{vmatrix}$$

(pull out α only from the 1st row)

$$= \alpha \cdot \alpha \cdot \begin{vmatrix} a_{11} & \cdots & a_{1n} \\ a_{21} & \cdots & a_{2n} \\ \alpha \cdot a_{31} & \cdots & \alpha \cdot a_{3n} \\ \cdots & \cdots & \cdots \\ \alpha \cdot a_{n1} & & \alpha \cdot a_{nn} \end{vmatrix}$$

(pull out α from the 2nd raw)

Inductively, one can pull out α from all the raw

Therefore,

$$\begin{vmatrix} \alpha a_{11} & \cdots & \alpha \cdot a_{1n} \\ \vdots & & \\ \alpha a_{n1} & \cdots & \alpha \cdot a_{nn} \end{vmatrix} = \underbrace{\alpha \alpha \cdots \cdots \alpha}_{n \text{ copies}} \cdot \begin{vmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \\ a_{n1} & \cdots & a_{nn} \end{vmatrix}$$

$$= \alpha^n \cdot \det A .$$

Proof 2: $\alpha \cdot A = \alpha I \cdot A$

$$= \begin{bmatrix} \alpha & & 0 \\ & \ddots & \\ 0 & & \alpha \end{bmatrix} \cdot A$$

$$\det(\alpha A) = \underbrace{\det \begin{bmatrix} \alpha & & 0 \\ & \ddots & \\ 0 & & \alpha \end{bmatrix}}_{\text{diagonal}} \cdot \det A$$

$$= \underbrace{\alpha \cdot \alpha \cdots \cdots \alpha}_{\text{(all the entries on the diagonal)}} \cdot \det A$$

$= \alpha^n \cdot \det A.$

6. Let A be a nonsingular matrix. Show that

5pt

$$\det(A^{-1}) = \frac{1}{\det(A)}$$

Proof. $A \cdot A^{-1} = I.$

$$\Rightarrow \det(A \cdot A^{-1}) = \det I = \det\begin{bmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{bmatrix} = 1$$

$$\Rightarrow \det(A) \cdot \det(A^{-1}) = 1$$

$$\Rightarrow \det(A^{-1}) = \frac{1}{\det A}.$$

7. Let A and B be 3×3 matrices with $\det(A) = 4$ and $\det(B) = 5$. Find the value of

8pt

(a) $\det(AB)$ **(b)** $\det(3A)$

(c) $\det(2AB)$ **(d)** $\det(A^{-1}B)$

(a). 2pt. $\det(AB) = \det(A) \cdot \det(B)$ 1pt/2

2pt: $\det(AB) = \det(A)\cdot\det(B)$ 1pt/2

$= 4\cdot 5 = 20$ 1pt/2

(b). $\det(3A) = 3^3 \det(A)$ 1pt/2

2pt.

$= 27\cdot 4$ 1pt/2

$= 108$

(c). $\det(2AB) = 2^3\cdot\det A\cdot\det B$ 1pt/2

2pt

$= 8\cdot 4\cdot 5$ 1pt/2

$= 160$

(d) $\det(A^{-1}B) = \det(A^{-1})\cdot\det B$

2pt

$= \frac{1}{\det A}\cdot\det B$ 1pt/2

$= \frac{1}{4}\cdot 5$ 1pt/2

$= \frac{5}{4}$

$$= \frac{5}{4}.$$