HW4

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Sec 1.4, 12, 13, 15;

来自 <https://users.math.msu.edu/users/zhangshiwen/s19/homework.html>

12. Let

3pt

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$

Show that if $d = a_{11}a_{22} - a_{21}a_{12} \neq 0$, then

$$A^{-1} = \frac{1}{d}\begin{bmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{bmatrix}$$

Proof.

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \cdot \frac{1}{d}\begin{bmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{bmatrix} = \frac{1}{d}\begin{bmatrix} a_{11}\cdot a_{22} - a_{12}\cdot a_{21} & 0 \\ 0 & a_{11}\cdot a_{22} - a_{12}\cdot a_{21} \end{bmatrix}$$

1pt/4

$$= \frac{1}{d}\begin{bmatrix} d & 0 \\ 0 & d \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$$

1pt/4

$$\frac{1}{d}\begin{bmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{bmatrix} \cdot \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} = \frac{1}{d}\begin{bmatrix} a_{22}\cdot a_{11} - a_{12}\cdot a_{21} & 0 \\ 0 & -a_{21}\cdot a_{12} + a_{11}\cdot a_{22} \end{bmatrix}$$

1pt/4

$$= \frac{1}{d}\begin{bmatrix} d & 0 \\ 0 & d \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$= I.$ 1pt/4.

therefore, $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$ is invertible

with the explicit inverse matrix given by $\frac{1}{d} \begin{bmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{bmatrix}$.

6pt. **13.** Use the result from Exercise 12 to find the inverse of each of the following matrices:

(a) $\begin{bmatrix} 7 & 2 \\ 3 & 1 \end{bmatrix}$ **(b)** $\begin{bmatrix} 3 & 5 \\ 2 & 3 \end{bmatrix}$ **(c)** $\begin{bmatrix} 4 & 3 \\ 2 & 2 \end{bmatrix}$

2pt (a). $d = 7 \cdot 1 - 3 \cdot 2 = 1$ 1pt/2

$\begin{bmatrix} 7 & 2 \\ 3 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} 1 & -2 \\ -3 & 7 \end{bmatrix}$ 1pt/2

2pt (b). $d = 3 \cdot 3 - 2 \cdot 5 = -1$ 1pt/2

$\begin{bmatrix} 3 & 5 \\ 2 & 3 \end{bmatrix}^{-1} = \underline{\frac{1}{-1} \cdot \begin{bmatrix} 3 & -5 \\ -2 & 3 \end{bmatrix}} = \begin{bmatrix} -3 & 5 \\ 2 & -3 \end{bmatrix}$ 1pt/2

lose 1 pt if the answer stops at this step.

2pt (c). $d = 4 \cdot 2 - 2 \cdot 3 = 2.$ 1pt/2

$\begin{bmatrix} 4 & 3 \\ 2 & 2 \end{bmatrix}^{-1} = \frac{1}{2} \cdot \begin{bmatrix} 2 & -3 \\ -2 & 4 \end{bmatrix} = \begin{bmatrix} 1 & -\frac{3}{2} \\ -1 & 2 \end{bmatrix}$ 1pt/2.

4pt **15.** Let A be a nonsingular matrix. Show that A^{-1} is also nonsingular and $(A^{-1})^{-1} = A$.

Proof. A is non-singular with inverse A^{-1}.

By the definition of inverse, we have.

$$A \cdot A^{-1} = A^{-1} \cdot A = I.$$

2pt/4

Therefore,

$$A^{-1} \cdot A = A \cdot A^{-1} = I,$$

which means A is also the inverse of A^{-1} and $(A^{-1})^{-1} = A$. 2pt/4.

Sec 1.5, 1, 2, 3, 5, 7

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4pt **1.** Which of the matrices that follow are elementary matrices? Classify each elementary matrix by type.

(a) $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ **(b)** $\begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}$

(c) $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 5 & 0 & 1 \end{bmatrix}$ **(d)** $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

(a). Type ① (or Type I) 1pt

(b). Not elementary matrix. 1pt

(c). Type ③ (or Type III) 1pt

(c). Type ③ (or Type III) 1pt

(d). Type ② (II). 1pt

(full credits if they describe the specific row operation).

4pt/**2.** Find the inverse of each matrix in Exercise 1. For each elementary matrix, verify that its inverse is an elementary matrix of the same type.

(a) $E^{-1} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ 1pt also type ①

(b) $E^{-1} = \begin{bmatrix} 2^{-1} & 0 \\ 0 & 3^{-1} \end{bmatrix}$ 1pt

(c). $E^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -5 & 0 & 1 \end{bmatrix}$ 1pt type ③

(d) $E^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 5^{-1} & 0 \\ 0 & 0 & 1 \end{bmatrix}$ 1pt type ②

(Only correct inverse is enough).

6pt **3.** For each of the following pairs of matrices, find an elementary matrix E such that $EA = B$.

(a) $A = \begin{bmatrix} 2 & -1 \\ 5 & 3 \end{bmatrix}$, $B = \begin{bmatrix} -4 & 2 \\ 5 & 3 \end{bmatrix}$

(b) $A = \begin{bmatrix} 2 & 1 & 3 \\ -2 & 4 & 5 \\ 3 & 1 & 4 \end{bmatrix}$, $B = \begin{bmatrix} 2 & 1 & 3 \\ 3 & 1 & 4 \\ -2 & 4 & 5 \end{bmatrix}$

(c) $A = \begin{bmatrix} 4 & -2 & 3 \\ 1 & 0 & 2 \\ -2 & 3 & 1 \end{bmatrix}$, $B = \begin{bmatrix} 4 & -2 & 3 \\ 1 & 0 & 2 \\ 0 & 3 & 5 \end{bmatrix}$

(a) $E = \begin{pmatrix} -2 & 0 \\ 0 & 1 \end{pmatrix}$ 2pt

Then $EA = B$

(multiplying the first row of A by -2 to get B).

(b) $E = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$ 2pt.

Then $EA = B$

(interchaing the 2nd row and the 3rd row of A to obtain B)

(c) $E = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 2 & 1 \end{bmatrix}$ 2pt

$EA = B$

adding 2 times the 2nd row of A to the 3rd row to obtain B.

5. Let

6pt

$$A = \begin{bmatrix} 1 & 2 & 4 \\ 2 & 1 & 3 \\ 1 & 0 & 2 \end{bmatrix}, \qquad B = \begin{bmatrix} 1 & 2 & 4 \\ 2 & 1 & 3 \\ 2 & 2 & 6 \end{bmatrix},$$

$$C = \begin{bmatrix} 1 & 2 & 4 \\ 0 & -1 & -3 \\ 2 & 2 & 6 \end{bmatrix}$$

(a) Find an elementary matrix E such that $EA = B$.

(b) Find an elementary matrix F such that $FB = C$.

(c) Is C row equivalent to A? Explain.

(a). $E = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$. Then $EA = B$.

2pt

(adding the 1st row to the 3rd row).

(b). $F = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}$, Then $FB = C$

2pt

$$\begin{bmatrix} & & \\ 0 & 0 & 1 \end{bmatrix}$$ 2pt

(subtracting the 3rd row from the 2nd row).

(c). By (a) and (b), we have

$$C = FB = F(EA) = (FE) \cdot A$$

2pt.

F and E are elementary matrices

Therefore, C is row equivalent to A.

8pt **7.** Let

$$A = \begin{bmatrix} 2 & 1 \\ 6 & 4 \end{bmatrix}$$

(a) Express A^{-1} as a product of elementary matrices.

(b) Express A as a product of elementary matrices.

(a) 4pt

$$\left[\begin{array}{cc|cc} 2 & 1 & 1 & 0 \\ 6 & 4 & 0 & 1 \end{array}\right] \xrightarrow{E_1 = \begin{bmatrix} 1 & 0 \\ -3 & 1 \end{bmatrix}}$$ 1pt/4

$$\left[\begin{array}{cc|cc} 2 & 1 & 1 & 0 \\ 0 & 1 & -3 & 1 \end{array}\right] \xrightarrow{E_2 = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}}$$ 1pt/4

$$\left[\begin{array}{cc|cc} 0 & 1 & -3 & 1 \end{array}\right] \longrightarrow$$

$$\left[\begin{array}{cc|cc} 2 & 0 & 4 & -1 \\ 0 & 1 & -3 & 1 \end{array}\right] \xrightarrow{E_3 = \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & 1 \end{bmatrix}}$$ 1pt/4.

$$\left[\begin{array}{cc|cc} 1 & 0 & 2 & -\frac{1}{2} \\ 0 & 1 & -3 & 1 \end{array}\right]$$

Therefore,

$$A^{-1} = \begin{bmatrix} 2 & -\frac{1}{2} \\ -3 & 1 \end{bmatrix} = E_3 \cdot E_2 \cdot E_1 = \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 \\ -3 & 1 \end{bmatrix}$$

1pt/4

(Writing $A^{-1} = \begin{bmatrix} 2 & -\frac{1}{2} \\ -3 & 1 \end{bmatrix}$ without any work on row op only gets 1 pt).

(The answer may not be unique, please check if A^{-1} is correct).

(b). 4 pt

$$A = (A^{-1})^{-1} = (E_3 \cdot E_2 \cdot E_1)^{-1}$$

$$= E_1^{-1} \cdot E_2^{-1} \cdot E_3^{-1}$$ 1pt

$$= \begin{bmatrix} 1 & 0 \\ -3 & 1 \end{bmatrix}^{-1} \cdot \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}^{-1} \cdot \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & 1 \end{bmatrix}^{-1}$$

$$= \begin{bmatrix} 1 & 0 \\ 3 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}$$

1pt 1pt 1pt

(full credits if they have different but correct answer of (a) and find the correct inverse of the elementary matrix).