HW12

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Sec 5.5, 1, 2, 4, 7(assuming V is R^3)

来自 <https://users.math.msu.edu/users/zhangshiwen/s19/homework.html>

1. Which of the following sets of vectors form an orthonormal basis for $\mathbb{R}^2$?

(a) $\{(1,0)^T, (0,1)^T\}$

(b) $\left\{\left(\frac{3}{5}, \frac{4}{5}\right)^T, \left(\frac{5}{13}, \frac{12}{13}\right)^T\right\}$

(c) $\{(1,-1)^T, (1,1)^T\}$

(d) $\left\{\left(\frac{\sqrt{3}}{2}, \frac{1}{2}\right)^T, \left(-\frac{1}{2}, \frac{\sqrt{3}}{2}\right)^T\right\}$

(a) Yes

$(1,0)\cdot\begin{pmatrix}0\\1\end{pmatrix} = 0$, $\Rightarrow (1,0)^T, (0,1)^T$ are orthogonal

$\left\|\begin{pmatrix}1\\0\end{pmatrix}\right\| = 1$, $\left\|\begin{pmatrix}0\\1\end{pmatrix}\right\| = 1$

$(1,0)^T, (0,1)^T$ form an orthonormal basis for $\mathbb{R}^2$

(b). No

$\left(\frac{3}{5}, \frac{4}{5}\right)\cdot\begin{pmatrix}\frac{5}{13}\\ \frac{12}{13}\end{pmatrix} \neq 0$. not orthogonal.

$\frac{1}{\sqrt{3}}$

(c). No.

$$\left\|\begin{pmatrix}1\\-1\end{pmatrix}\right\| = \sqrt{1^2+(-1)^2} = \sqrt{2} \neq 1$$ Not unit vector.

(d). Yes.

$$\left(\frac{\sqrt{3}}{2}, \frac{1}{2}\right)\cdot\begin{pmatrix}-\frac{1}{2}\\ \frac{\sqrt{3}}{2}\end{pmatrix} = \frac{\sqrt{3}}{2}\cdot\left(-\frac{1}{2}\right) + \frac{1}{2}\cdot\frac{\sqrt{3}}{2} = 0$$, orthogonal.

$$\left\|\left(\frac{\sqrt{3}}{2}, \frac{1}{2}\right)^T\right\| = \sqrt{\left(\frac{\sqrt{3}}{2}\right)^2+\left(\frac{1}{2}\right)^2} = \sqrt{\frac{3}{4}+\frac{1}{4}} = 1$$

$$\left\|\left(-\frac{1}{2}, \frac{\sqrt{3}}{2}\right)^T\right\| = \sqrt{\left(-\frac{1}{2}\right)^2+\left(\frac{\sqrt{3}}{2}\right)^2} = \sqrt{\frac{1}{4}+\frac{3}{4}} = 1$$

2. Let

$$\mathbf{u}_1 = \begin{bmatrix} \frac{1}{3\sqrt{2}} \\ \frac{1}{3\sqrt{2}} \\ -\frac{4}{3\sqrt{2}} \end{bmatrix}, \quad \mathbf{u}_2 = \begin{bmatrix} \frac{2}{3} \\ \frac{2}{3} \\ \frac{1}{3} \end{bmatrix}, \quad \mathbf{u}_3 = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \\ 0 \end{bmatrix}$$

(a) Show that $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$ is an orthonormal basis for $\mathbb{R}^3$.

$$\langle u_1, u_2\rangle = \frac{1}{3\sqrt{2}}\cdot\frac{2}{3} + \frac{1}{3\sqrt{2}}\cdot\frac{2}{3} - \frac{4}{3\sqrt{2}}\cdot\frac{1}{3}$$

$$= \frac{2}{9\sqrt{2}} + \frac{2}{9\sqrt{2}} - \frac{4}{9\sqrt{2}} = 0.$$

$$\langle u_1, u_3 \rangle = \frac{1}{3\sqrt{2}} \cdot \frac{1}{\sqrt{2}} + \frac{1}{3\sqrt{2}} \cdot \left(-\frac{1}{\sqrt{2}}\right) + 0$$

$$= \frac{1}{6} - \frac{1}{6} = 0$$

$$\langle u_2, u_3 \rangle = \frac{2}{3} \cdot \frac{1}{\sqrt{2}} + \frac{2}{3} \cdot \left(-\frac{1}{\sqrt{2}}\right) + 0$$

$$= 0.$$

$$\|u_1\| = \sqrt{\left(\frac{1}{3\sqrt{2}}\right)^2 + \left(\frac{1}{3\sqrt{2}}\right)^2 + \left(-\frac{4}{3\sqrt{2}}\right)^2}$$

$$= \sqrt{\frac{1}{18} + \frac{1}{18} + \frac{16}{18}} = \sqrt{\frac{18}{18}} = 1.$$

$$\|u_2\| = \sqrt{\left(\frac{2}{3}\right)^2 + \left(\frac{2}{3}\right)^2 + \left(\frac{1}{3}\right)^2}$$

$$= \sqrt{\frac{4}{9} + \frac{4}{9} + \frac{1}{9}} = \sqrt{\frac{9}{9}} = 1.$$

$$\|u_3\| = \sqrt{\left(\frac{1}{\sqrt{2}}\right)^2 + \left(-\frac{1}{\sqrt{2}}\right)^2 + 0^2}$$

$$= \sqrt{\frac{1}{2} + \frac{1}{2} + 0} = 1$$

$$= \sqrt{\frac{1}{2} + \frac{1}{2} + 0} = 1.$$

(b) Let $\mathbf{x} = (1, 1, 1)^T$. Write $\mathbf{x}$ as a linear combination of $\mathbf{u}_1$, $\mathbf{u}_2$, and $\mathbf{u}_3$ using Theorem 5.5.2 and use Parseval's formula to compute $\|\mathbf{x}\|$.

$$\overline{x} = c_1 u_1 + c_2 \cdot u_2 + c_3 \cdot u_3$$

$$c_1 = \langle \overline{x}, u_1 \rangle$$

$$= (1, 1, 1) \cdot \begin{pmatrix} \frac{1}{3\sqrt{2}} \\ \frac{1}{3\sqrt{2}} \\ -\frac{4}{3\sqrt{2}} \end{pmatrix} = \frac{1}{3\sqrt{2}} + \frac{1}{3\sqrt{2}} - \frac{4}{3\sqrt{2}}$$

$$= \frac{-2}{3\sqrt{2}}.$$

$$c_2 = \langle \overline{x}, u_2 \rangle$$

$$= (1, 1, 1) \cdot \begin{pmatrix} \frac{2}{3} \\ \frac{2}{3} \\ \frac{1}{3} \end{pmatrix} = \frac{2}{3} + \frac{2}{3} + \frac{1}{3} = \frac{5}{3}.$$

$$c_3 = \langle \overline{x}, u_3 \rangle$$

$$= (1, 1, 1) \cdot \begin{pmatrix} \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \\ \end{pmatrix} = \frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}} + 0 = 0.$$

$$\left|\begin{matrix}\frac{\ }{\sqrt{2}}\\ 0\end{matrix}\right) = \frac{\ }{\sqrt{2}} - \frac{\ }{\sqrt{2}} + 0 = 0.$$

Therefore, $\overline{x} = \frac{-2}{3\sqrt{2}} \cdot u_1 + \frac{5}{3} \cdot u_2$

$$\|\overline{x}\|^2 = 1^2 + 1^2 + 1^2 = 3, \quad c_1^2 + c_2^2 + c_3^2 = \left(\frac{-2}{3\sqrt{2}}\right)^2 + \left(\frac{5}{3}\right)^2 = \frac{4}{9 \cdot 2} + \frac{25}{9} = \frac{27}{9} = 3$$

4. Let θ be a fixed real number and let

$$\mathbf{x}_1 = \begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix} \quad \text{and} \quad \mathbf{x}_2 = \begin{bmatrix} -\sin\theta \\ \cos\theta \end{bmatrix}$$

(a) Show that $\{\mathbf{x}_1, \mathbf{x}_2\}$ is an orthonormal basis for $\mathbb{R}^2$.

(b) Given a vector $\mathbf{y}$ in $\mathbb{R}^2$, write it as a linear combination $c_1\mathbf{x}_1 + c_2\mathbf{x}_2$.

(c) Verify that

$$c_1^2 + c_2^2 = \|\mathbf{y}\|^2 = y_1^2 + y_2^2$$

(a) $$\langle \overline{x}_1, \overline{x}_2 \rangle = (\cos\theta, \sin\theta) \cdot \begin{pmatrix} -\sin\theta \\ \cos\theta \end{pmatrix}$$

$$= \cos\theta \cdot (-\sin\theta) + \sin\theta \cdot \cos\theta = 0.$$

$$\|\overline{x}_1\| = \sqrt{\cos^2\theta + \sin^2\theta} = \sqrt{1} = 1$$

$$\|\overline{x}_2\| = \sqrt{(-\sin\theta)^2 + (\cos\theta)^2} = \sqrt{1} = 1.$$

(b). $\overline{y} = (y_1, y_2)^T.$

$$c_1 = \langle \bar{y}, \bar{x}_1 \rangle = y_1 \cdot \cos\theta + y_2 \cdot \sin\theta$$

$$c_2 = \langle \bar{y}, \bar{x}_2 \rangle = y_1 \cdot (-\sin\theta) + y_2 \cdot \cos\theta .$$

$$\bar{y} = (y_1 \cos\theta + y_2 \sin\theta) \cdot \bar{x}_1 + (-y_1 \sin\theta + y_2 \cos\theta) \bar{x}_2$$

$$c_1^2 + c_2^2 = (y_1 \cdot \cos\theta + y_2 \sin\theta)^2 + (-y_1 \cdot \sin\theta + y_2 \cos\theta)^2$$

$$= y_1^2 \cdot \cos^2\theta + 2y_1 y_2 \cos\theta \sin\theta + y_2^2 \cdot \sin^2\theta$$

$$+ (-y_1 \sin\theta)^2 - 2y_1 y_2 \sin\theta \cos\theta + (y_2 \cdot \cos\theta)^2$$

$$= y_1^2 (\cos^2\theta + \sin^2\theta) + 0 + y_2^2 (\sin^2\theta + \cos^2\theta)$$

$$= y_1^2 + y_2^2 = \|\bar{y}\|^2$$

7. Let $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$ be an orthonormal basis for an inner product space V. If $\mathbf{x} = c_1\mathbf{u}_1 + c_2\mathbf{u}_2 + c_3\mathbf{u}_3$ is a vector with the properties $\|\mathbf{x}\| = 5$, $\langle \mathbf{u}_1, \mathbf{x} \rangle = 4$, and $\mathbf{x} \perp \mathbf{u}_2$, then what are the possible values of c_1, c_2, c_3?

$$c_1 = \langle \bar{x}, u_1 \rangle = \langle u_1, \bar{x} \rangle = 4$$

$c_1 = \langle \bar{x}, u_1 \rangle = \langle u_1, \bar{x} \rangle = 4$

$c_2 = \langle \bar{x}, u_2 \rangle = 0$ since $\bar{x} \perp u_2$.

$\|\bar{x}\|^2 = 5^2 = c_1^2 + c_2^2 + c_3^2 = 4^2 + 0^2 + c_3^2$

$\Rightarrow c_3^2 = 25 - 16 = 9$

$\Rightarrow c_3 = \pm 3$.

Sec 5.6, 3, 8

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3. Given the basis $\{(1,2,-2)^T, (4,3,2)^T, (1,2,1)^T\}$ for $\mathbb{R}^3$, use the Gram–Schmidt process to obtain an orthonormal basis.

$\bar{x}_1 = (1,2,-2)^T$, $\bar{x}_2 = (4,3,2)^T$, $\bar{x}_3 = (1,2,1)^T$

$u_1 = \frac{1}{\|\bar{x}_1\|} \cdot \bar{x}_1 = \frac{1}{\sqrt{1+4+4}} \cdot (1,2,-2)^T$

$= \frac{1}{3}(1,2,-2)^T = (\frac{1}{3}, \frac{2}{3}, \frac{-2}{3})^T$

$\langle x_2, u_1 \rangle = (4,3,2) \cdot (\frac{1}{3}, \frac{2}{3}, \frac{-2}{3})^T$

$= \frac{4}{3} + 2 - \frac{4}{3} = 2$

$$= \frac{4}{3} + 2 - \frac{4}{3} = 2.$$

$$P_1 = \langle X_2, u_1\rangle \cdot u_1 = 2\left(\frac{1}{3}, \frac{2}{3}, -\frac{2}{3}\right)^T$$

$$= \left(\frac{2}{3}, \frac{4}{3}, -\frac{4}{3}\right)^T.$$

$$V_2 = X_2 - P_1$$

$$= (4, 3, 2)^T - \left(\frac{2}{3}, \frac{4}{3}, -\frac{4}{3}\right)^T$$

$$= \left(\frac{10}{3}, \frac{5}{3}, \frac{10}{3}\right)^T.$$

$$\|V_2\| = \sqrt{\left(\frac{10}{3}\right)^2 + \left(\frac{5}{3}\right)^2 + \left(\frac{10}{3}\right)^2}$$

$$= \sqrt{\frac{100 + 25 + 100}{9}} = \sqrt{\frac{225}{9}}$$

$$= \sqrt{25}$$

$$= 5.$$

$$u_2 = \frac{1}{\|V_2\|} \cdot V_2 = \frac{1}{5}\left(\frac{10}{3}, \frac{5}{3}, \frac{10}{3}\right)^T$$

$$= \left(\frac{2}{3}, \frac{1}{3}, \frac{2}{3}\right)^T.$$

$$\langle X_3, u_1\rangle = (1, 2, 1) \cdot \left(\frac{1}{3}, \frac{2}{3}, -\frac{2}{3}\right)^T$$

$$\langle \overline{x_3}, u_1 \rangle = (1,2,1)\cdot\left(\tfrac{1}{3}, \tfrac{2}{3}, -\tfrac{2}{3}\right)^T$$

$$= \tfrac{1}{3} + \tfrac{4}{3} - \tfrac{2}{3} = 1$$

$$\langle \overline{x_3}, u_2 \rangle = (1,2,1)\left(\tfrac{2}{3}, \tfrac{1}{3}, \tfrac{2}{3}\right)^T$$

$$= \tfrac{2}{3} + \tfrac{2}{3} + \tfrac{2}{3} = 2.$$

$$\overline{p_2} = \langle \overline{x_3}, u_1 \rangle \cdot u_1 + \langle \overline{x_3}, u_2 \rangle \cdot u_2$$

$$= 1\cdot\left(\tfrac{1}{3}, \tfrac{2}{3}, -\tfrac{2}{3}\right)^T + 2\left(\tfrac{2}{3}, \tfrac{1}{3}, \tfrac{2}{3}\right)^T$$

$$= \left(\tfrac{5}{3}, \tfrac{4}{3}, \tfrac{2}{3}\right)^T$$

$$v_3 = \overline{x_3} - \overline{p_2}$$

$$= (1,2,1)^T - \left(\tfrac{5}{3}, \tfrac{4}{3}, \tfrac{2}{3}\right)^T$$

$$= \left(-\tfrac{2}{3}, \tfrac{2}{3}, \tfrac{1}{3}\right)^T$$

$$\|v_3\| = \sqrt{\left(-\tfrac{2}{3}\right)^2 + \left(\tfrac{2}{3}\right)^2 + \left(\tfrac{1}{3}\right)^2}$$

$$= \sqrt{4+4+1}$$

$$= \sqrt{\frac{4+4+1}{9}} = 1.$$

$$u_3 = v_3 = (-\frac{2}{3}, \frac{2}{3}, \frac{1}{3})^T$$

8. Use the Gram–Schmidt process to find an orthonormal basis for the subspace of $\mathbb{R}^4$ spanned by $\mathbf{x}_1 = (4, 2, 2, 1)^T$, $\mathbf{x}_2 = (2, 0, 0, 2)^T$, and $\mathbf{x}_3 = (1, 1, -1, 1)^T$.

$$\|\bar{x}_1\| = \sqrt{4^2 + 2^2 + 2^2 + 1^2} = \sqrt{16+4+4+1}$$
$$= \sqrt{25} = 5.$$

$$u_1 = \frac{1}{5}\bar{x}_1 = (\frac{4}{5}, \frac{2}{5}, \frac{2}{5}, \frac{1}{5})^T.$$

$$\langle \bar{x}_2, u_1 \rangle = (2, 0, 0, 2)(\frac{4}{5}, \frac{2}{5}, \frac{2}{5}, \frac{1}{5})^T$$
$$= \frac{8}{5} + \frac{2}{5} = 2$$

$$p_1 = \langle \bar{x}_2, u_1 \rangle \cdot u_1 = 2u_1 = (\frac{8}{5}, \frac{4}{5}, \frac{4}{5}, \frac{2}{5})^T.$$

$$V_2 = \bar{x}_2 - \bar{p}_1 = (2, 0, 0, 2)^T - (\frac{8}{5}, \frac{4}{5}, \frac{4}{5}, \frac{2}{5})^T$$
$$= (\frac{2}{5}, \frac{-4}{5}, \frac{-4}{5}, \frac{8}{5})^T.$$

$$\|V_2\| = \sqrt{(\frac{2}{5})^2 + (-\frac{4}{5})^2 + (-\frac{4}{5})^2 + (\frac{8}{5})^2}$$
$$= \sqrt{\frac{4+16+16+64}{25}}$$
$$= \sqrt{\frac{100}{25}} = \sqrt{4} = 2$$

$$= \sqrt{\frac{100}{25}} = \sqrt{4} = 2.$$

$$u_2 = \frac{1}{2} v_2 = \left(\frac{1}{5}, \frac{-2}{5}, \frac{-2}{5}, \frac{4}{5}\right)^T$$

$$\langle x_3, u_1 \rangle = (1, 1, -1, 1)\left(\frac{4}{5}, \frac{2}{5}, \frac{2}{5}, \frac{1}{5}\right)^T$$

$$= \frac{4}{5} + \frac{2}{5} - \frac{2}{5} + \frac{1}{5} = 1.$$

$$\langle x_3, u_2 \rangle = (1, 1, -1, 1) \cdot \left(\frac{1}{5}, \frac{-2}{5}, \frac{-2}{5}, \frac{4}{5}\right)^T$$

$$= \frac{1}{5} - \frac{2}{5} + \frac{2}{5} + \frac{4}{5} = 1.$$

$$p_2 = \langle x_3, u_1 \rangle \cdot u_1 + \langle x_3, u_2 \rangle \cdot u_2$$

$$= u_1 + u_2$$

$$= \left(\frac{4}{5}, \frac{2}{5}, \frac{2}{5}, \frac{1}{5}\right)^T + \left(\frac{1}{5}, \frac{-2}{5}, \frac{-2}{5}, \frac{4}{5}\right)^T$$

$$= (1, 0, 0, 1)^T.$$

$$v_3 = x_3 - p_2 = (1, 1, -1, 1)^T - (1, 0, 0, 1)^T$$

$$= (0, 1, -1, 0)^T$$

$$\|v_3\| = \sqrt{2}$$

$$u_3 = \frac{1}{\sqrt{2}} v_3 = \frac{1}{\sqrt{2}}(0, 1, -1, 0)^T$$