HW11

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- **Sec 5.3, 1(a), 3(a).**
- **Sec 5.4, 1, 2, 15(a)(b)**

来自 <https://users.math.msu.edu/users/zhangshiwen/s19/homework.html>

1. Find the least squares solution of each of the following systems:

8pt **(a)** $x_1 + x_2 = 3$, $2x_1 - 3x_2 = 1$, $0x_1 + 0x_2 = 2$ **(b)** $-x_1 + x_2 = 10$, $2x_1 + x_2 = 5$, $x_1 - 2x_2 = 20$

$$A = \begin{bmatrix} 1 & 1 \\ 2 & -3 \\ 0 & 0 \end{bmatrix}, \quad b = \begin{bmatrix} 3 \\ 1 \\ 2 \end{bmatrix}.$$

$$A^T \cdot A = \begin{bmatrix} 1 & 2 & 0 \\ 1 & -3 & 0 \end{bmatrix} \cdot \begin{bmatrix} 1 & 1 \\ 2 & -3 \\ 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 5 & -5 \\ -5 & 10 \end{bmatrix}.$$ 2pt/8

$$A^T \cdot b = \begin{bmatrix} 1 & 2 & 0 \\ 1 & -3 & 0 \end{bmatrix} \begin{bmatrix} 3 \\ 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 5 \\ 0 \end{bmatrix}$$ 2pt/8

$$A^T \cdot A x = A^T \cdot b$$

$$\left[\begin{array}{cc:c} 5 & -5 & 5 \\ -5 & 10 & 0 \end{array}\right] \rightarrow \left[\begin{array}{cc|c} 1 & -1 & 1 \\ -1 & 2 & 0 \end{array}\right]$$

$$\rightarrow \left[\begin{array}{cc:c} 1 & -1 & 1 \\ 0 & 1 & 1 \end{array}\right]$$

$$\begin{cases} x_1 - x_2 = 1 \\ x_2 = 1 \end{cases} \Rightarrow \begin{cases} x_1 = 2 \\ x_2 = 1 \end{cases}$$

4pt/8

least squares solution: $(2,1)^T$.

8pt **3.** For each of the following systems $A\mathbf{x} = \mathbf{b}$, find all least squares solutions:

(a) $A = \begin{bmatrix} 1 & 2 \\ 2 & 4 \\ -1 & -2 \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}$

$$A^T \cdot A = \begin{bmatrix} 1 & 2 & -1 \\ 2 & 4 & -2 \end{bmatrix} \cdot \begin{bmatrix} 1 & 2 \\ 2 & 4 \\ -1 & -2 \end{bmatrix}$$

$$= \begin{bmatrix} 6 & 12 \\ 12 & 24 \end{bmatrix}$$

2pt/8

$$A^T\bar{b} = \begin{bmatrix} 1 & 2 & -1 \\ 2 & 4 & -2 \end{bmatrix} \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 6 \\ 12 \end{bmatrix}$$ 2pt/8

$$A^T A \bar{x} = A^T \bar{b}$$

$$\left[\begin{array}{cc:c} 6 & 12 & 6 \\ 12 & 24 & 12 \end{array}\right] \rightarrow \left[\begin{array}{cc:c} 1 & 2 & 1 \\ 0 & 0 & 0 \end{array}\right]$$

$x_1 + 2x_2 = 1.$ $\quad x_1 = -2x_2 + 1.$

$x_2 = \alpha,\ x_1 = -2\alpha + 1.$

All least squares solutions:

$$\begin{pmatrix} -2\alpha+1 \\ \alpha \end{pmatrix},$$

4pt/8

8pt **1.** Let $\mathbf{x} = (-1, -1, 1, 1)^T$ and $\mathbf{y} = (1, 1, 5, -3)^T$. Show that $\mathbf{x} \perp \mathbf{y}$. Calculate $\|\mathbf{x}\|_2$, $\|\mathbf{y}\|_2$, $\|\mathbf{x}+\mathbf{y}\|_2$ and verify that the Pythagorean law holds.

$<\bar{x}, \bar{y}> = -1 - 1 + 5 - 3 = 0 \Rightarrow \bar{x} \perp \bar{y}$ 2pt/8

$\|\bar{x}\|_2 = \sqrt{(-1)^2 + (-1)^2 + 1^2 + 1^2} = \sqrt{4} = 2$ 2pt/8

$\|\bar{y}\|_2 = \sqrt{1^2 + 1^2 + 5^2 + (3)^2} = \sqrt{36} = 6$ 2pt/8

$\bar{x}+\bar{y} = (0,0,6,-2)^T$

$\|\bar{x}+\bar{y}\|_2 = \sqrt{0^2+0^2+6^2+(-2)^2} = \sqrt{40}$ 2pt/8

$\|\bar{x}\|_2^2 + \|\bar{y}\|_2^2 = 4+36 = 40 = \|\bar{x}+\bar{y}\|_2^2$.

14pt **2.** Let $\mathbf{x} = (1, 1, 1, 1)^T$ and $\mathbf{y} = (8, 2, 2, 0)^T$.

(a) Determine the angle θ between $\mathbf{x}$ and $\mathbf{y}$.

(b) Find the vector projection $\mathbf{p}$ of $\mathbf{x}$ onto $\mathbf{y}$.

(c) Verify that $\mathbf{x} - \mathbf{p}$ is orthogonal to $\mathbf{p}$.

(d) Compute $\|\mathbf{x} - \mathbf{p}\|_2$, $\|\mathbf{p}\|_2$, $\|\mathbf{x}\|_2$ and verify that the Pythagorean law is satisfied.

3pt (a)

$$\cos\theta = \frac{\langle \bar{x}, \bar{y} \rangle}{\|\bar{x}\| \cdot \|\bar{y}\|} = \frac{8+2+2}{\sqrt{4} \cdot \sqrt{64+4+4}} = \frac{12}{2 \cdot \sqrt{72}} = \frac{12}{2 \cdot 6 \cdot \sqrt{2}} = \frac{\sqrt{2}}{2}$$ 3pt

$\theta = \frac{\pi}{4}$.

3pt (b). $\bar{p} = \frac{\langle \bar{x}, \bar{y} \rangle}{\|\bar{y}\|^2} \cdot \bar{y} = \frac{12}{\ldots}(8, 2, 2, 0)^T$

3pt (b). $\bar{p} = \frac{\langle \bar{x}, \bar{y} \rangle}{\|\bar{y}\|^2} \cdot \bar{y} = \frac{12}{72}(8,2,2,0)^T$

$= \frac{1}{6}(8,2,2,0)^T$ 3pt

4pt (c). $\bar{x} - \bar{p} = (1,1,1,1)^T - (\frac{4}{3}, \frac{1}{3}, \frac{1}{3}, 0)^T$

$= (\frac{-1}{3}, \frac{2}{3}, \frac{2}{3}, 1)^T.$ 2pt/4

$\langle \bar{x} - \bar{p}, \bar{p} \rangle = (-\frac{1}{3}, \frac{2}{3}, \frac{2}{3}, 1) \cdot \frac{1}{6}(8,2,2,0)^T$

$= \frac{1}{6}(-\frac{8}{3} + \frac{4}{3} + \frac{4}{3} + 0) = 0.$ 2pt/4.

$\Rightarrow \bar{x} - \bar{p} \perp \bar{p}.$

4pt (d). $\|\bar{x} - \bar{p}\|_2 = \sqrt{\frac{1}{9} + \frac{4}{9} + \frac{4}{9} + 1} = \sqrt{2}$ 2pt/4

$\|\bar{p}\|_2 = \sqrt{\frac{1}{36}(64+4+4)} = \sqrt{2}.$ 2pt/4

$\|\bar{x} - \bar{p}\|_2^2 + \|\bar{p}\|_2^2 = 2 + 2 = 4 = \|\bar{x}\|_2^2$

15. Compute $\|\mathbf{x}\|_1$, $\|\mathbf{x}\|_2$, and $\|\mathbf{x}\|_\infty$ for each of the following vectors in $\mathbb{R}^3$.

(a) $\mathbf{x} = (-3, 4, 0)^T$ **(b)** $\mathbf{x} = (-1, -1, 2)^T$

6 (a) $\|\vec{x}\|_1 = 3 + 4 = 7$ 2pt

$\|\vec{x}\|_2 = \sqrt{(-3)^2 + 4^2} = \sqrt{25} = 5.$ 2pt

$\|\vec{x}\|_\infty = \max(|-3|, 4, 0) = 4.$ 2pt

6. (b) $\|\vec{x}\|_1 = 1 + 1 + 2 = 4.$ 2pt

$\|\vec{x}\|_2 = \sqrt{(-1)^2 + (-1)^2 + 2^2} = \sqrt{6}$ 2pt

$\|\vec{x}\|_\infty = 2.$ 2pt