§11.7. Strategy for Testing Series

1. Three basic types of series and one rule.

① p-Series. $\sum \frac{1}{n^p}$ < ConV if $p>1$; DIV if $p \le 1$

② Geometric Series: $\sum a \cdot r^{n-1}$ (or $\sum a \cdot r^n$, $\sum a \cdot r^{n+1}$, etc) < ConV if $|r|<1$ $\left(= \frac{a}{1-r}\right)$; DIV if $|r| \ge 1$

③ Alternating Series: $\sum (-1)^n \cdot b_n$ (or $\sum (-1)^{n-1} b_n$, $\sum (-1)^{n+1} b_n$, etc) < ConV if $\lim_n b_n = 0$ and b_n decreases. DIV if $\lim b_n \ne 0$.

One rule: n^{th} term test for divergence.

④ If $\lim a_n \ne 0$, then $\sum a_n$ is divergent.

2. Check whether ①-④ can be applied directly or can be applied through some simple simplification (of a_n).

e.g. Test for ConV/DIV.

- $\sum 3 \cdot n^{-\frac{3}{2}}$ p-Series $p=\frac{3}{2}$, ConV
- $\sum \frac{3^n}{2^{2n}}$ G.S. $r=\frac{3}{4}$, ConV.
- $\sum (-1)^{n+1} \frac{1}{\sqrt{n}}$ A.S. $b_n = \frac{1}{\sqrt{n}}$ ConV
- $\sum (-1)^n \cdot e^{\frac{1}{n}}$ DIV Test, $\lim_n e^{\frac{1}{n}} = e^0 = 1 \ne 0$.

- $\sum \left(5^n + \frac{1}{n^2}\right) = \sum 5^n + \sum \frac{1}{n^2}$ DIV + ConV ⇒ DIV
- $\sum \frac{n+1}{n^3} = \sum \frac{1}{n^2} + \sum \frac{1}{n^3}$ ConV + ConV ⇒ ConV.
- $\sum \frac{3^n + 2^n}{4^n} = \sum \left(\frac{3}{4}\right)^n + \sum \left(\frac{2}{4}\right)^n$ ConV + ConV ⇒ ConV.

3. If none of ①-④ works, then consider whether a_n can be compared with p-Series or G.S. And try to apply (limit) Comparison Test.

- $\sum \frac{1-6n+3n^2}{n^4+1}$ compare with $\sum \frac{3n^2}{n^4}$. ConV.
- $\sum \frac{2^n+1}{3^n-1}$ compare with $\sum \frac{2^n}{3^n}$ ConV. via Limit Comparison Test $\lim_{n\to\infty} \frac{a_n}{b_n} = 1$.

4. If a_n is not so clear to be compared with some known series, then try Ratio Test. In particularly, if a_n contains factorial $n!$ or the problem asks to test ABS convergence, then consider Ratio Test first.

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = L \begin{cases} L<1, & \sum a_n \text{ is ABS convergent and also convergent} \\ L>1, & \sum a_n \text{ is divergent} \\ L=1. & \text{R. Test inconclusive. (Then try Alternating S. Test or Definition of ABS conv)} \end{cases}$$

e.g. $a_n = \dfrac{3^n \cdot n^2}{n!}$, $\left|\dfrac{a_{n+1}}{a_n}\right| = \dfrac{3^{n+1}\cdot(n+1)^2}{(n+1)!}\cdot\dfrac{n!}{3^n\cdot n^2} = \dfrac{3^{n+1}}{3^n}\cdot\dfrac{(n+1)^2}{n^2}\cdot\dfrac{n!}{(n+1)!} = 3\cdot\dfrac{(n+1)^2}{n^2}\cdot\dfrac{1}{n+1} = \dfrac{3(n+1)}{n^2}$

$\Rightarrow \lim_{n\to\infty}\left|\dfrac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\dfrac{3(n+1)}{n^2} = 0\ (<1)$. $\Rightarrow \sum \dfrac{3^n\cdot n^2}{n!}$ conv.

5. If 2-4 do not work, then try Integral Test. In particularly, for series $\sum_{n=2}^{\infty}\dfrac{1}{(\ln n)^p\cdot n}$, use Integral Test with $\int_2^{\infty}\dfrac{1}{(\ln x)^p\cdot x}dx$ (p can be positive or negative)

e.g. $\sum_{n=2}^{\infty}\dfrac{1}{\sqrt{\ln n}\cdot n} \overset{\text{I. Test}}{\sim} \int_2^{\infty}\dfrac{1}{\sqrt{\ln x}\cdot x}dx = \lim_{t\to\infty}\int_2^t \dfrac{1}{\sqrt{\ln x}\cdot x}dx$, $\quad u=\ln x$, $du = \frac{1}{x}dx$.

DIV since $\int_2^{\infty}\dfrac{1}{\sqrt{\ln x}\cdot x}dx = \infty$. $\quad = \lim_{t\to\infty}\int \dfrac{1}{\sqrt{u}}du = \lim_{t\to\infty} 2\sqrt{u}\Big| = \lim_{t\to\infty} 2\sqrt{\ln x}\Big|_2^t$

$= \lim_{t\to\infty} 2\sqrt{\ln t} - 2\sqrt{\ln 2} = \infty$

6. Conclusion:

Order to chase tests: $\left.\begin{array}{l}\text{DIV Test}\\ p\text{-Series}\\ \text{Geo. Series}\\ \text{Alternating}\end{array}\right\} \longrightarrow$ (limit) Comparison $\rightarrow$ Ratio Test $\rightarrow$ Integral Test.

Remark 1: Above tests are used to determine whether $\sum a_n$ is convergent or DIV. They can not be used to compute the exact value of $\sum a_n$ EXCEPT Geometric Series (Test). In other words, once you see the key words "Find the sum" "Compute the series" etc, try to use the formula $\sum_{n=0}^{\infty} a\cdot r^n = \dfrac{a}{1-r}$, $|r|<1$

Remark 2: If a_n are all positive (or all negative), ABS conv and conv are the same. They are different only for alternating series $\sum(-1)^n\cdot b_n$ and ABS conv $\Rightarrow$ conv. (Not vice verse)

§11.8 Power Series:

Motivation: The standard Geometric Sum for $1+x+x^2+x^3+\cdots$

$$\sum_{n=0}^{\infty} x^n \begin{cases} \frac{1}{1-x} & \text{conv for } |x|<1. \\ \text{DIV} & \text{for } |x|\geq 1. \end{cases}$$

Previously, $\sum a_n$'s nth term a_n are considered $\boxed{\text{FIXED NUMBERS}}$. In the rest of the chapter, we want to consider $\sum a_n$ and a_n as $\boxed{\text{function of } x.}$ Like above, $a_n = x^n$.

Power Series:

$$\sum_{n=0}^{\infty} c_n\cdot(x-a)^n = c_0 + c_1\cdot(x-a)^1 + c_2\cdot(x-a)^2 + c_3\cdot(x-a)^3 + \cdots + c_n\cdot(x-a)^n + \cdots\cdots$$

where a is a constant, $c_0, c_1, c_2, \cdots$ is a sequence. This series is a function of x, called Power Series. (each term is a power function of x). We also call a the center and c_n the coefficients.

e.g. 1. Find the center and coefficients of the following power series.

$$\sum_{n=0}^{\infty} x^n = \sum_{n=0}^{\infty} 1\cdot(x-0)^n, \quad a=0, \quad c_0=c_1=c_2=\cdots=1 \quad (c_n = 1 \text{ for all } n).$$

(S16) $\displaystyle\sum_{k=0}^{\infty} \frac{(-1)^k\cdot x^k}{3^k\cdot(k+1)}$, $\quad a=0, \quad c_k = \dfrac{(-1)^k}{3^k\cdot(k+1)}, \; k=0,1,2,\cdots$

★ (S15) $\displaystyle\sum_{n=1}^{\infty} \frac{(2x+1)^n}{n}$ $\qquad \dfrac{(2x+1)^n}{n} = \dfrac{[2(x+\frac{1}{2})]^n}{n} = \dfrac{2^n}{n}\cdot\left[x-(-\tfrac{1}{2})\right]^n$, $\boxed{a=-\frac{1}{2}}$, $c_n = \boxed{\dfrac{2^n}{n}}$

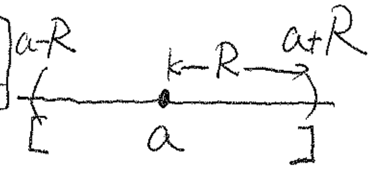
Remark: center a can be found by letting $2x+1=0 \Rightarrow x=-\frac{1}{2}$.

★ (f14) $-3+9(x+5)-27\cdot(x+5)^2+81\cdot(x+5)^3-\cdots\cdots$

$$= \sum_{n=0}^{\infty}(-3)^{n+1}\cdot(x+5)^n, \quad a=-5, \quad c_n=(-3)^{n+1}, \; n=0,1,2,\cdots\cdots$$

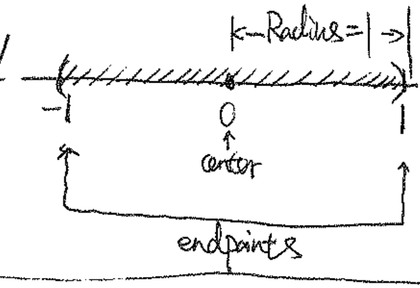
Goal: We want to know FOR WHICH VALUES (of x) does $\sum c_n\cdot(x-a)^n$ converge?

Method: Let $a_n = c_n\cdot(x-a)^n$ and apply Ratio Test to determine the $\boxed{\text{RADIUS of Convergence } R}$ and $\boxed{\text{INTERVAL of Convergence}}$



e.g.2 (Trivial example) $\sum_{n=0}^{\infty} x^n$. According to standard Geometric Series, $\sum_{n=0}^{\infty} x^n$ is $\boxed{\text{convergent if } |x|<1}$ and divergent if $|x| \geq 1$.

$|x|<1$ represents $-1<x<1$, i.e., the following interval



Conclusion: Radius of Conv: $R=1$.
Interval of Conv: $(-1, 1)$ (both sides are open).

(Non-Trivial examples by ratio test)

e.g.3 Find the radius of convergence of $\sum_{k=0}^{\infty} \frac{(-1)^k \cdot x^k}{3^k (k+1)}$

(5/6, M.C)

Solution: Let $a_k = \frac{(-1)^k \cdot x^k}{3^k \cdot (k+1)}$ and apply (full version) Ratio Test.

$$a_{k+1} = \frac{(-1)^{k+1} \cdot x^{k+1}}{3^{k+1} \cdot (k+2)}, \quad \left|\frac{a_{k+1}}{a_k}\right| = \left| \frac{\frac{(-1)^{k+1} \cdot x^{k+1}}{3^{k+1} \cdot (k+2)}}{\frac{(-1)^k \cdot x^k}{3^k \cdot (k+1)}} \right| = \left| \frac{(-1)^{k+1} \cdot x^{k+1}}{3^{k+1} \cdot (k+2)} \cdot \frac{3^k \cdot (k+1)}{(-1)^k \cdot x^k} \right|$$

$$= \left| \frac{(-1)^{k+1}}{(-1)^k} \cdot \frac{x^{k+1}}{x^k} \cdot \frac{3^k (k+1)}{3^{k+1} (k+2)} \right| = \left| (-1) \cdot x \cdot \frac{k+1}{3(k+2)} \right| = \frac{k+1}{3(k+2)} \cdot |x|.$$

$\boxed{\text{Caution:}}$ ☆☆☆ x can be both positive and negative. DO NOT drop the abstract value for x.

$$\lim_{k\to\infty} \left|\frac{a_{k+1}}{a_k}\right| = \lim_{k\to\infty} \frac{k+1}{3(k+2)} \cdot |x| = \frac{1}{3} |x|$$

(x is fixed, limit does not effect x).

☆ $\boxed{\text{THEN SET ABOVE LIMIT } < 1}$, i.e., $\boxed{\frac{1}{3}|x| < 1} \Rightarrow \boxed{|x| < 3}$

$\boxed{\text{Radius of Conv } R = 3}$.

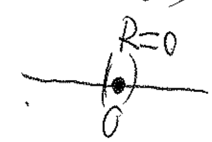
Radius of Conv.

Remark: Radius of Convergence might be $R=0$ or $R=+\infty$.

e.g.4. Find radius of Conv for the following power series

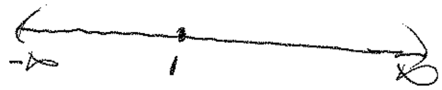
- $\sum n!(2x)^n$, $a_n = n!(2x)^n$, $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left|\frac{(n+1)! (2x)^{n+1}}{n! (2x)^n}\right| = \lim_{n\to\infty} |(n+1) \cdot (2x)| = \infty$ (unless $x=0$)

Except $x=0$, for all other values of x, $\lim = \infty > 1$. $\boxed{R=0}$



- $\sum \frac{(x-1)^n}{n!}$, $a_n = \frac{(x-1)^n}{n!}$, $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left| \frac{(x-1)^{n+1}}{(n+1)!} \cdot \frac{n!}{(x-1)^n} \right| = \lim_{n\to\infty} \left|\frac{(x-1)}{n+1}\right| = 0$ (for all x)

The limit $= 0 < 1$ $\boxed{\text{for all } x}$. $\boxed{R=\infty}$



☆ e.g.5. (Radius + INTERVAL).

(S15). 14pts. Find $\boxed{\text{all values of } x}$ for which the series $\sum_{n=1}^{\infty} \frac{(2x+1)^n}{n}$ converges.

Solution: • (Step 1: Ratio Test for Radius of Conv)

$$a_n = \frac{(2x+1)^n}{n}, \quad \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{(2x+1)^{n+1}}{(n+1)} \cdot \frac{n}{(2x+1)^n}\right| = \lim_{n\to\infty}\left|(2x+1)\cdot\frac{n}{n+1}\right| = |2x+1|$$ (KEEP the abstract value)

Set the limit to be less than 1. $|2x+1| < 1$ ($R = \frac{1}{2}$ since $|x+\frac{1}{2}| < \frac{1}{2}$).

Solve the inequality for x. $\boxed{-1 < 2x+1 < 1}$ Caution: Two sided inequality when Abs is removed.

i.e. $-2 < 2x < 0 \Rightarrow \boxed{-1 < x < 0}$, $x \in (-1, 0)$

Two endpoints are $x=-1$, $x=0$.

• (Step 2: Test endpoints)

at $x = 0$, $\sum_{n=1}^{\infty} \frac{(2x+1)^n}{n} \overset{x=0}{=\!=} \sum_{n=1}^{\infty} \frac{(1)^n}{n} = \sum_{n=1}^{\infty} \frac{1}{n}$, p-Series, $p=1$, DIV.

$\sum_{n=1}^{\infty} \frac{(2x+1)^n}{n}$ is DIV at $x=0$, therefore, $x=0$ is NOT inclued (open).

at $x=-1$, $\sum_{n=1}^{\infty} \frac{(2x+1)^n}{n} \overset{x=-1}{=\!=} \sum_{n=1}^{\infty} \frac{(-2+1)^n}{n} = \sum_{n=1}^{\infty} (-1)^n \frac{1}{n}$, alternating series with $b_n = \frac{1}{n}$.

$\lim_{n\to\infty} b_n = \lim \frac{1}{n} = 0$ and $\frac{1}{n}$ is decreasing, therefore, $\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$ is convergent.

$\sum_{n=1}^{\infty} \frac{(2x+1)^n}{n}$ is Conv at $x=-1$, therefore, $x=-1$ is inclued (closed)

So the series $\sum_{n=1}^{\infty} \frac{(2x+1)^n}{n}$ converges on $[-1, 0)$, (or converges for $x \in [-1, 0)$)

(or converges for $-1 \le x < 0$)

Conclusion: Find all values of x for which $\sum c_n (x-a)^n$ converges.

Step 1: Set $a_n = c_n \cdot (x-a)^n$. Compute $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = L < 1$

Step 2: Set limit $L < 1$, we have such inequality $c|x-a| < 1 \Leftrightarrow |x-a| < \boxed{\frac{1}{c} = R}$

Solve for x, we have an interval $(a - \frac{1}{c}, a + \frac{1}{c})$

Step 3: Test the two endpoints $x = a - \frac{1}{c}$ and $x = a + \frac{1}{c}$ (separately).

Usually, one endpoint will need A.S. Test.

More examples

e.g 6 (§17) Consider the function of the power series $\sum_{n=1}^{\infty} \frac{(3x-2)^n}{n \cdot 2^{n+1}}$

(a). Find the open interval of convergence for the power series.

(b) Test the right/~~left~~ endpoints for CONV/DIV

(a) $a_n = \frac{(3x-2)^n}{n \cdot 2^{n+1}}$, $a_{n+1} = \frac{(3x-2)^{n+1}}{(n+1) \cdot 2^{n+2}}$, $\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{(3x-2)^{n+1}}{(n+1) \cdot 2^{n+2}} \cdot \frac{n \cdot 2^{n+1}}{(3x-2)^n}\right|$

$$= \frac{n}{(n+1) \cdot 2} \cdot |3x-2|$$

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \frac{n}{(n+1) 2} \cdot |3x-2| = \frac{|3x-2|}{2}$$

Let $\left|\frac{3x-2}{2}\right| < 1$, solve for x. $|3x-2| < 2 \iff -2 < 3x-2 < 2$

$\iff 0 < 3x < 4$

$\iff \boxed{0 < x < \frac{4}{3}}$

(open) Interval of conv: $(0, \frac{4}{3})$

Right endpoint: $x = \frac{4}{3}$ Plug into $\sum_{n=1}^{\infty} \frac{(3x-2)^n}{n \cdot 2^{n+1}} = \sum_{n=1}^{\infty} \frac{(3 \cdot \frac{4}{3} - 2)^n}{n \cdot 2^{n+1}} = \sum_{n=1}^{\infty} \frac{1}{2n}$ DIV.

left endpoint: $x=0$ Plug into $\sum_{n=1}^{\infty} \frac{(3x-2)^n}{n \cdot 2^{n+1}} = \sum_{n=1}^{\infty} \frac{(3 \cdot 0 - 2)^n}{n \cdot 2^{n+1}} = \sum_{n=1}^{\infty} \frac{(-2)^n}{n \cdot 2^{n+1}}$

Conv due to A.S.T. $= \sum_{n=1}^{\infty} \frac{(-1)^n \cdot 2^n}{n \cdot 2^{n+1}} = \sum_{n=1}^{\infty} \frac{(-1)^n}{2n}$.

Remark 1: The inequality: $|A \cdot x + B| < C$ leads to two-sided inequalities when removing absolute value: $-C < Ax + B < C$

Remark 2: For $\sum_{n=0}^{\infty} C_n (x-a)^n$, a direct formula for Radius of Conv is

$$R = \frac{1}{\lim_{n\to\infty} \left|\frac{C_{n+1}}{C_n}\right|}$$ (Can be derived by Ratio Test, not required)

Remark 3: If $R=0$, then the "open" interval of conv is a single point $x=a$.

If $R=\infty$, then the interval of conv is $(-\infty, \infty)$ (any x in $\mathbb{R}$)

§11.9 Representation of Functions as Power Series

Basic Formula: $1+x+x^2+\cdots = \boxed{\sum_{n=0}^{\infty} x^n = \frac{1}{1-x}}$, $|x|<1$.

Rewrite as:

$$\star \quad \boxed{\frac{1}{1-\blacksquare} = \sum_{n=0}^{\infty} \blacksquare^n \quad \text{for } |\blacksquare|<1}$$

Goal: Want to use the above formula to REPRENT a FUNCTION as a POWER SERIES. and FIND its RADIUS of CONVERGENCE.

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e.g.1 (S16, 15pts) Consider the function $g(x)=\frac{5}{1-4x^2}$. (a) Express g as a power series in sigma-notation. (b). What is the radius of convergence for this series.

solution: (a). $g(x)=5\cdot\frac{1}{1-\boxed{4x^2}}$ apply $\star$ with $\blacksquare = 4x^2$

$$= 5\cdot\sum_{n=0}^{\infty}(4x^2)^n = \boxed{\sum_{n=0}^{\infty} 5\cdot(4x^2)^n} = \sum_{n=0}^{\infty} \underbrace{5\cdot 4^n}_{C_n}\cdot x^{2n}$$

(full credits at this step).

(b). The series converges if $|\blacksquare|<1$, i.e.,

$|4x^2|<1$ (keep in mind that Power Series is essentially a Geometric series which converges when $|r|<1$)

$\Rightarrow x^2<\frac{1}{4} \Rightarrow |x|<\sqrt{\frac{1}{4}}=\boxed{\frac{1}{2}}$

Therefore, the radius of convergence is $\boxed{\frac{1}{2}}$ ※.

Remarks:

① The problem may ask you to find the first several non-zero terms. (Let's say 3 for ex),

1st non-zero term $=5$, 2nd non-zero term $=5\cdot4\cdot x^2$, 3rd $=5\cdot4^2\cdot x^4$

since $\sum_{n=0}^{\infty}5\cdot(4x^2)^n = 5\cdot(4x^2)^0+5\cdot(4x^2)^1+5\cdot(4x^2)^2+\cdots$

② It may also ask you "If the series is in $\sum_{n=0}^{\infty} C_n\cdot x^{2n}$ form, what is C_n?"

$\boxed{C_n=5\cdot4^n}$, $n=0,1,2,\cdots$. In ①, we $\boxed{\text{include } x}$ and in ② we donot.

eg. 2 Represent $\frac{3x}{3+x^2}$ as a power series and find its radius of conv.
(s/b, MC).

Remark: The crucial step is to find $\boxed{1}$ and $\boxed{\blacksquare}$ in the denominator $\frac{1}{1-\blacksquare}$
(create)

$$3+x^2 = 3\cdot\left[1+\frac{x^2}{3}\right] = 3\cdot\left[1-\left(-\frac{x^2}{3}\right)\right]$$

It is important to have a negative sign here.

$\boxed{\text{Solution}}$: $\frac{3x}{3+x^2} = 3x\cdot\frac{1}{3+x^2} = 3x\cdot\frac{1}{3\cdot\left[1-\left(-\frac{x^2}{3}\right)\right]}$

$$= 3x\cdot\frac{1}{3}\cdot\frac{1}{1-\left(-\frac{x^2}{3}\right)}$$

Hint: $\blacksquare = -\frac{x^2}{3} = -\frac{1}{3}\cdot x^2$

$$(\dagger) \quad = x\cdot\sum_{n=0}^{\infty}\left(-\frac{x^2}{3}\right)^n$$

$|\blacksquare| < 1$.

$$= \sum_{n=0}^{\infty} x\cdot\left(-\frac{1}{3}\right)^n\cdot(x^2)^n = \sum_{n=0}^{\infty}\left(-\frac{1}{3}\right)^n\cdot x^{1+2n}$$

Radius of Conv: $\boxed{\left|-\frac{x^2}{3}\right|<1} \Rightarrow |x^2|<3 \Rightarrow |x|<\boxed{\sqrt{3}}$. $\boxed{R=\sqrt{3}}$

Remarks: ① If the problem only ask you to find the first several terms in the sum, then you can expand the sum at $(\dagger)$, ie.

$$x\cdot\sum_{n=0}^{\infty}\left(-\frac{x^2}{3}\right)^n = x\cdot\left[\left(-\frac{x^2}{3}\right)^0+\left(-\frac{x^2}{3}\right)^1+\left(-\frac{x^2}{3}\right)^2+\dots\right]$$

$$= x\cdot\left[1-\frac{x^2}{3}+\frac{x^4}{9}-\dots\right]$$

$$= \boxed{x-\frac{x^3}{3}+\frac{x^5}{9}-\dots}$$

② $\sqrt{x^2}=|x|$. Do not forget the absolute value when you reduce inequality $x^2<a$ to $|x|<\sqrt{a}$, and the open interval of conv is $(-\sqrt{a},\sqrt{a})$. (You will not be asked to test endpoints for Representation Prob)

③ You need to make the constant part to 1 not the x part. The following is

× × × × × × ×
WRONG ATTEPT: $\frac{3x}{x^2+3} = \frac{3x}{x^2\cdot\left[1+\frac{3}{x^2}\right]} = \frac{3}{x}\cdot\frac{1}{1-\left(\frac{-3}{x^2}\right)} = \frac{3}{x}\cdot\sum_{n=0}^{\infty}\left(\frac{-3}{x^2}\right)^n$
× × × × × × ×

we need positive power of x.

eg 3. Find the power series and its radius of conv for $f(x)=\frac{x^2}{2+x}$.
(§15)

Solution: $f(x)=x^2\cdot\frac{1}{2+x}=x^2\cdot\frac{1}{2\left[1-\left(-\frac{x}{2}\right)\right]}$

$$=\frac{x^2}{2}\cdot\sum_{n=0}^{\infty}\left(-\frac{x}{2}\right)^n$$

$$=\sum_{n=0}^{\infty}\frac{x^2}{2}\cdot\frac{(-1)^n\cdot x^n}{2^n}=\boxed{\sum_{n=0}^{\infty}\frac{(-1)^n}{2^{n+1}}\cdot x^{n+2}}$$

for $\left|-\frac{x}{2}\right|<1$, i.e. $|x|<\boxed{2}$, radius of conv $R=2$, $C_n=\frac{(-1)^n}{2^{n+1}}$

★★★ (Advanced Topics) Differential and integration of Power Series (related to ww 6 and 7). If $f(x)=\sum_{n=0}^{\infty}C_n\cdot x^n$, then

$$f'(x)=\left(\sum_{n=0}^{\infty}C_n\cdot x^n\right)'=\sum_{n=0}^{\infty}(C_n\cdot x^n)'=\sum_{n=1}^{\infty}C_n\cdot n\cdot x^{n-1}\text{ and}$$

$$\int f(x)dx=\int\sum_{n=0}^{\infty}C_n\cdot x^n dx=\sum_{n=0}^{\infty}\int C_n x^n dx=\sum_{n=0}^{\infty}C_n\cdot\frac{1}{n+1}\cdot x^{n+1}$$

for $|x|<1$.

eg 4. Express $g(x)=\frac{3}{(1-x)^3}$ as power series and find its radius of convergence
(§14).

$$\frac{1}{1-x}=\sum_{n=0}^{\infty}x^n$$

Take derivatives both sides: $\frac{-1}{(1-x)^2}\cdot(-1)=\left(\sum_{n=0}^{\infty}x^n\right)'=\sum_{n=1}^{\infty}n\cdot x^{n-1}$ ($n=0$ the first term is constant derivative is 0)

Take derivative again: $\left(\frac{1}{(1-x)^2}\right)'=\left(\sum_{n=1}^{\infty}n\cdot x^{n-1}\right)'$

chain rule:
left hand side: $\frac{-2}{(1-x)^3}\cdot(-1)=\sum_{n=2}^{\infty}n\cdot(n-1)\cdot x^{n-2}$

Then $g(x)=\frac{3}{(1-x)^3}=\frac{3}{2}\cdot\boxed{\frac{2}{(1-x)^3}}=\frac{3}{2}\cdot\sum_{n=2}^{\infty}n(n-1)\cdot x^{n-2}=\sum_{n=2}^{\infty}\frac{3}{2}n(n-1)\cdot x^{n-2}$

for $|x|<1$.

eg 5. Represent $f(x) = \tan^{-1}(7x)$ as power series and find
(ww 7) its radius of convergence

Hint: $(\tan^{-1} x)' = \frac{1}{1+x^2}$ where we have representation for $\frac{1}{1+x^2}$.

Solution: $f'(x) = \frac{7}{1+(7x)^2} = \frac{7}{1-[-(7x)^2]} = 7 \cdot \sum_{n=0}^{\infty} [-(7x)^2]^n$

$$(+) \qquad = 7 \cdot \sum_{n=0}^{\infty} (-1)^n \cdot 7^{2n} \cdot x^{2n}$$

Then $f(x) = \int f'(x)\,dx = \int \cancel{7} \sum_{n=0}^{\infty} (-1)^n \cdot 7^{2n+1} \cdot x^{2n} \cdot dx$

$$= \sum_{n=0}^{\infty} \int (-1)^n \cdot 7^{2n+1} \cdot x^{2n}\,dx.$$

$$= \sum_{n=0}^{\infty} (-1)^n \cdot 7^{2n+1} \cdot \int x^{2n}\,dx$$

$$= \sum_{n=0}^{\infty} (-1)^n \cdot 7^{2n+1} \frac{x^{2n+1}}{2n+1}$$

Radius of conv: $|-(7x)^2| < 1 \Rightarrow (7x)^2 < 1$

$R = \frac{1}{7}$. $\qquad \Rightarrow |7x| < 1$

$$\Rightarrow \boxed{|x| < \frac{1}{7}}$$

Remark: In the webwork, you do not need to integrate the whole series (all terms). You only need to integrate the first three terms in (+), ie.

$$f'(x) = 7 \cdot \sum_{n=0}^{\infty} (-1)^n \cdot 7^{2n} \cdot x^{2n} = 7 \cdot (1 - 7^2 \cdot x^2 + 7^4 \cdot x^4 + \dots) = 7 - 7^3 \cdot x^2 + 7^5 \cdot x^4 + \dots$$

$$f(x) = \int (7 - 7^3 x^2 + 7^5 \cdot x^4 + \dots)\,dx = 7x - 7^3 \cdot \frac{1}{3} x^3 + 7^5 \cdot \frac{1}{5} \cdot x^5 + \dots$$