§11.1 Sequences

Definition: A sequence is A LIST of numbers, denoted by $a_1, a_2, a_3, \cdots, a_n, \cdots$

a_n is the nth term of the sequence $(n=1,2,3,\cdots)$

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eg0 (Motivation) Investment at bank: \$1000 is deposited into bank at 6% annual interest rate. Find the money in the account at the end of the nth year.

1 year, $1000 + 1000 \times 6\% = 1000 \times 1.06$ $\quad a_1$

2 years, $1000 \times 1.06 + 1000 \times 1.06 \times 6\% = 1000 \times 1.06^2$ $\quad a_2$

......

n years $\cdots\cdots = 1000 \times \underbrace{1.06 \times 1.06 \times \cdots \times 1.06}_{n} = \boxed{1000 \times 1.06^n \quad a_n}$

Goals of this section.

I: Find a formula of a_n given the first several terms of the sequence

II: Find the limit of a sequence a_n. (Determine whether a_n is CONV or DIV).

Remarks:

① $a_1, a_2, \cdots, a_n, \cdots$ is also denoted by $\{a_n\}_{n=1}^{\infty}$, $\{a_n\}$, or $\{a_1, \cdots, a_n, \cdots\}$ for short

② Sequence Vs Function. a_n vs $f(x)$. Replace n in the formula of a_n by x to generate a function $f(x)$ in the usual sense and vice verse.

$a_n \rightleftarrows f(x)$. eg. $a_n = n \cdot e^{-n} \longleftrightarrow f(x) = x \cdot e^{-x}$

③ LIST vs Collection. a_n has a definite order, labeled by the index $n=1,2,\cdots$

We have 'the first term' a_1, '2nd term' a_2, $\cdots$, '100th term' a_{100} etc

④ Trivial sequence: Constant sequence: All a_n in the list have the same value

eg. $\{a_n = 0\}_{n=0}^{\infty}$ represents the list: $0, 0, 0, \cdots, 0, \cdots$

I: Formula for a_n

e.g.1. Consider the following sequence: $9, 25, 49, 81, \cdots$

Write a formula for the nth term of the sequence, $n=1,2,3,\cdots$

Hint: $9=(2\cdot\underset{\substack{\triangle\\ n=1}}{1}+1)^2$, $25=(2\cdot\underset{\substack{\triangle\\ n=2}}{2}+1)^2$, $49=(2\cdot\underset{\substack{\triangle\\ n=3}}{3}+1)^2$, $81=(2\cdot\underset{\substack{\triangle\\ n=4}}{4}+1)^2$

Solution: $a_n=(2n+1)^2$, $n=1,2,3,\cdots$

e.g. 2. The sequence $-1, +1, -1, +1, \cdots$ has formula $a_n=(-1)^n$, $n=1,2,\cdots$

The sequence $+1, -1, +1, -1, \cdots$ has formula $a_n=(-1)^{n+1}$, $n=1,2,\cdots$

☆ e.g. 3 Find the formula for the sequence $\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \cdots$

and for $-1, \frac{1}{2}, -\frac{1}{4}, \frac{1}{8}, -\frac{1}{16}, \cdots$

Hint: $2^0=1$, $2^1=2$, $2^2=4$, $2^3=8$, $2^4=16$, $\cdots$

First sequence: $a_n=\frac{1}{2^n}$, $n=1,2,3,4,\cdots$

Second sequence: $a_n=(-1)^n\cdot\frac{1}{2^{n-1}}$, $n=1,2,3,4,\cdots$

Rmk: Remember to double check the answer by plugging in $n=1,2,3,4,\cdots$

$(-1)^n\cdot\frac{1}{2^{n-1}}$ can be re-written as $(-1)^n\cdot\frac{1}{2^{n-1}}=-\left(-\frac{1}{2}\right)^{n-1}$ or $2\cdot\left(-\frac{1}{2}\right)^n$

Some frequently used sequence:

Alternating sequence: $(-1)^n$, $(-1)^{n+1}$, $(-1)^{n-1}$

ODD/Even sequence: $a_n=2n-1$, $n=1,2,\cdots$. $a_n=2n$, $n=1,2,\cdots$

Exponential (Geometric) sequence: $a_n=2^n$, $a_n=\frac{1}{3^n}$, $a_n=(-\frac{4}{5})^n$

Polynomial (Decay) sequence: $a_n=\frac{1}{n}$, $a_n=\frac{1}{n^2}$, $a_n=\frac{1}{\sqrt{n}}$
(p-series)

II. Limit of a_n as $n \to \infty$

If $\lim_{n\to\infty} a_n = L$ (exists and is a finite number), we say a_n converges to L. Otherwise, a_n is divergent.

eg4. Constant sequence always converges to the constant. (Constant function)

$a_n = 5$, $n = 1, 2, \cdots$ $\boxed{\lim_{n\to\infty} a_n = 5 \quad \text{conV}}$

eg5. Consider the sequence $a_n = n \cdot e^{-n}$, $n = 1, 2, \cdots$ Find the limit of a_n

Rmk: The problem is equivallent to find $\lim_{x\to\infty} x \cdot e^{-x}$ as a function limit.

Solution: $\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{n}{e^n} = \frac{\infty}{\infty}$ (indeterminate form)

$\underline{\underline{l'H}} \lim_{n\to\infty} \frac{(n)'}{(e^n)'}$ derivatives with respect to n (as the variable)

$= \lim_{n\to\infty} \frac{1}{e^n} = \frac{1}{e^\infty} = \frac{1}{\infty} = \boxed{0}$ conV.

eg6 Evaluate

$\lim_{n\to\infty} \frac{-2n^3+1}{n^2+3n} \xlongequal[\text{rule}]{\text{leading term}} \lim_{n\to\infty} \frac{-2n^3}{n^2} = \lim_{n\to\infty} -2n = -\infty$. DIV

Some special limits:

- $a_n = (-1)^n$, $\{-1, 1, +1, -1, \cdots\}$. The sequence jumps between -1, $+1$.
  DOES NOT have a limit.

- $\lim_{n\to\infty} r^n = \begin{cases} +\infty & \text{if } r > 0 \\ 0 & \text{if } 0 < r < 1 \end{cases}$ $\quad \lim_{n\to\infty} 2^n = \infty \quad 2^\infty = \infty$

  $\lim_{n\to\infty} \left(\frac{1}{3}\right)^n = 0 \quad \left(\frac{1}{3}\right)^\infty = \frac{1}{3^\infty} = 0$

- $\lim_{n\to\infty} (-1)^n \cdot \frac{1}{2^{n-1}} = 0$, since $\frac{1}{2^{n-1}} \to \frac{1}{2^\infty} = 0$ while $(-1)^n$ is either $+1$ or -1

  $(\pm 1) \cdot 0 = 0$ (any finite number times $0 = 0$)

Review of some frequently used limit techniques from Cal I and §6.8 Cal II.

① $\boxed{\text{Leading Term Trick}}$ for $\frac{\text{Polynomial}}{\text{Polynomial}}$ type. (only n to some power is involved).

Rule: Find the terms with highest order (power) in n. Drop all the rest terms. Simplify the expression and then take the limit as $n \to \infty$.

$$\lim_{n\to\infty} \frac{6-2n+n^4}{5n^4-3n^2+100} = \lim_{n\to\infty} \frac{\cancel{6-2n}+n^4}{5n^4\cancel{-3n^2+100}}$$

← leading term in the numerator

leading term in the denominator.

$$= \lim_{n\to\infty} \frac{\cancel{n^4}}{5\cancel{n^4}} \overset{\text{simplify}}{=\!=} \boxed{\frac{1}{5}} \quad \text{conv}$$

$$\lim_{n\to\infty} \frac{-3\cdot n^2+5n}{1000\cdot n+5\cdot n^{2.5}+6} = \lim_{n\to\infty} \frac{-3n^2}{5\cdot n^{2.5}} = \lim_{n\to\infty} \frac{-3}{5n^{0.5}} = \lim_{n\to\infty} \frac{-3}{5\sqrt{n}} = \frac{-3}{5\cdot\infty} = \boxed{0}$$

Remark: Compare these two examples with $\boxed{\text{e.g. 6}}$ and the $\boxed{\text{l'Hospital rule}}$.

② $\boxed{a^{+\infty} = +\infty \text{ if } a>1}$ and $\boxed{a^{+\infty} = 0 \text{ if } 0<a<1}$.

Remember the limits via the following examples:

$$\lim_{n\to\infty} 2^{n-1} = 2^\infty = \infty, \quad \lim_{n\to\infty} [1.01]^n = 1.01^\infty = \infty; \quad \lim_{n\to\infty} \left(\tfrac{1}{3}\right)^n = \left(\tfrac{1}{3}\right)^\infty = 0$$

For negative numbers, use the trick $\boxed{(-a)^n = (-1)^n \cdot a^n}$. $(-5)^n = (-1)^n \cdot 5^n$

③ Exponential l'Hospital Trick (Please refer to lec-notes Week 4, Page 7, e.g. 4 in §6.8)
(WW11, 12 are relied on this)

Evaluate $\lim_{n\to\infty} 3\cdot\left(1+\frac{1}{n^2}\right)^{5n}$

It is enough to consider the limit for

$$\left(1+\frac{1}{n^2}\right)^{5n} = e^{\ln\left(1+\frac{1}{n^2}\right)^{5n}}$$

Step 1: $\ln\left(1+\frac{1}{n^2}\right)^{5n} = 5n\cdot\ln\left(1+\frac{1}{n^2}\right)$

$$= \frac{\ln\left(1+\frac{1}{n^2}\right)}{\frac{1}{5n}}$$

Step 2:

$$\lim_{n\to\infty} \frac{\ln\left(1+\frac{1}{n^2}\right)}{\frac{1}{5n}} \overset{\text{l'H}}{=} \lim_{n\to\infty} \frac{\frac{1}{1+\frac{1}{n^2}}\cdot\frac{-2}{n^3}}{\frac{-1}{5n^2}} = \lim_{n\to\infty} \frac{10\cdot n^2}{n^3+\cancel{n}} = 0 \quad \text{(leading term Rule)}$$

Step 3:

$$\lim_{n\to\infty} 3\left(1+\frac{1}{n^2}\right)^{5n} = 3\cdot\lim_{n\to\infty} e^{\boxed{\ln\left(1+\frac{1}{n^2}\right)^{5n}}} = 3\cdot e^{0} = \boxed{3}$$

← results from Step 2

§11.2 Series

Key points:

① Definition and properties of Series: $\sum_{n=1}^{\infty} a_n = a_1 + a_2 + \cdots + a_n + \cdots$

②. TEST FOR DIVERGENCE (DIV TEST). $\lim_{n\to\infty} a_n \neq 0$ implies $\sum_{n=1}^{\infty} a_n$ DIV.
(Also called nth ~~term~~ term test for divergence)

③ Geometric Series: $a + a\cdot r + a\cdot r^2 + a\cdot r^3 + \cdots + a\cdot r^{n-1} + \cdots = \frac{a}{1-r}$, $|r|<1$.

- Motivation: (Sigma Notation) (Remark: For a complete review of sigma notation, refer to Appendix E in the textbook).

eg.0. Expand the following SIGMA NOTATION as a usual sum.

$$\sum_{n=0}^{4} \frac{1}{2^n} = \frac{1}{2^0} + \frac{1}{2^1} + \frac{1}{2^2} + \frac{1}{2^3} + \frac{1}{2^4}\ ; \quad \sum_{n=2}^{5} \frac{1}{n-1} = \frac{1}{2-1} + \frac{1}{3-1} + \frac{1}{4-1} + \frac{1}{5-1}.$$

$$\sum_{i=1}^{N} a_i = a_1 + a_2 + \cdots + a_{N-2} + a_{N-1} + a_N\ ; \quad \sum_{k=1}^{100} 3 = \underbrace{3+3+\cdots+3+3}_{100}\ (= 3\cdot 100 = 300)$$

Question: what happens if we put ∞ on the top of Σ? (i.e. $\sum_{n=1}^{\infty}$)

- **Definition:** Give a sequence $\{a_n\}_{n=1}^{\infty}$ (i.e. $a_1, a_2, a_3, \cdots, a_n, \cdots$). We call the INFINITE SUM of all a_n an infinite SERIES: $a_1 + a_2 + a_3 + \cdots + a_n + \cdots$.
And denote by $\sum_{n=1}^{\infty} a_n = a_1 + a_2 + \cdots + a_n + \cdots$ for short.

- eg.1. (Two trivial examples)

$$\sum_{n=0}^{\infty} 0 = 0+0+0+\cdots+0+\cdots\ (=0)\ ; \quad \sum_{n=1}^{\infty} 3 = 3+3+\cdots+3+\cdots\ (=\infty, \text{ we may guess})$$

- We are interested in whether the infinite sum adds up to some finite number or infinity. Roughly speaking, if $\sum_{n=1}^{\infty} a_n$ adds up to some (fixed) finite number we say $\boxed{\text{the series is CONVERGENT}}$. Otherwise, $\boxed{\sum_{n=1}^{\infty} a_n \text{ diverges}}$.

- Remark: There are only FEW examples we can compute the exact sum of the series. All the rest of Chapter 11 are methods to determine CONV/DIV without knowing the exact sum.

- Linear properties of Series: $\{a_n\}_{n=1}^{\infty}$, $\{b_n\}_{n=1}^{\infty}$ are two sequences, C is a constant.

$$\sum_{n=1}^{\infty}(C\cdot a_n)=C\left(\sum_{n=1}^{\infty}a_n\right);\quad \sum_{n=1}^{\infty}(a_n+b_n)=\left(\sum_{n=1}^{\infty}a_n\right)+\left(\sum_{n=1}^{\infty}b_n\right);\ \sum_{n=1}^{\infty}a_n=a_1+\sum_{n=2}^{\infty}a_n=\cdots=a_1+a_2+a_3+\sum_{n=4}^{\infty}a_n=\cdots$$

- $\boxed{\text{TEST FOR DIVERGENCE. (DIV TEST)}}$ (Also called "n^{th} term test" for short)

Theorem: If $\lim_{n\to\infty} a_n \neq 0$, then $\sum_{n=1}^{\infty} a_n$ is DIVERGENT. i.e., if the sequence a_n does not have a limit $\boxed{\text{or}}$ the limit is NOT zero, then the corresponding series $\sum_{n=1}^{\infty} a_n$ is divergent.

e.g. 2. Fill in the table. Determine whether the following sequences and series are Conv or DIV.

a_n	$(-1)^n$	$2+\sin(\frac{1}{n})$	$\frac{2n+3}{2n+1}$	$\left(\frac{3}{2}\right)^{n-1}$
$\lim_{n\to\infty} a_n$.	D.N.E. (± 1)	$2+\sin 0=2$	$\frac{2}{2}=1$	∞ (since $\frac{3}{2}>1$)
a_n is Conv or DIV	DIV	CONV	CONV	DIV
$\sum_{n=1}^{\infty} a_n$ is Conv or DIV.	DIV	DIV	DIV	DIV

Remark: If $\lim_{n\to\infty} a_n = 0$, DIV TEST is inconclusive. We need to move on to methods in the following section

- One particular series (sequence) we can compute $\boxed{\text{the exact sum}}$.

Geometric Series. (G.S.): $a_n = a\cdot r^{n-1}$, $n=1,2,3,\cdots$. a, r are two constants

$a + a\cdot r + a\cdot r^2 + \cdots$, which can be written as $\boxed{\sum_{n=1}^{\infty} a\cdot r^{n-1} = \sum_{n=0}^{\infty} a\cdot r^n}$.

a is called the FIRST TERM of the series. r is called COMMON RATIO.

- Conclusion on CONV/DIV of G.S.

If $|r|\geq 1$, $\sum_{n=1}^{\infty} a\cdot r^{n-1}$ is divergent. If $|r|<1$, $\sum_{n=1}^{\infty} a\cdot r^{n-1}$ is convergent AND.

$$\boxed{\sum_{n=1}^{\infty} a\cdot r^{n-1} = \frac{a}{1-r}} \left(= \frac{\text{FIRST TERM}}{1-\text{COMMON RATIO}}\right)$$

eg.3. Consider $\sum_{n=1}^{\infty}\left(\frac{3}{2}\right)^{n-1}$ in eg.2. This is a G.S. with $a=1$, $r=\frac{3}{2}$.

$|r|=\frac{3}{2}>1 \Rightarrow$ The Geometric Series diverges.

eg.4 Find the sum of the series $\sum_{n=1}^{\infty}\frac{2^{n-1}-3^{n+1}}{4^n}$
(s17, MC).

Step 1: Break into two G.S.s (linear properties) $\sum_{n=1}^{\infty}\frac{2^{n-1}-3^{n+1}}{4^n}=\sum_{n=1}^{\infty}\frac{2^{n-1}}{4^n}-\sum_{n=1}^{\infty}\frac{3^{n+1}}{4^n}$

Step 2: G.S.1. $\sum_{n=1}^{\infty}\frac{2^{n-1}}{4^n}$ G.S. $a=\frac{1}{4}$, $r=\frac{2}{4}=\frac{1}{2}$

$=\frac{\frac{1}{4}}{1-\frac{1}{2}}=\boxed{\frac{1}{2}}$ Hint: $\frac{2^{n-1}}{4^n}=\frac{2^{n-1}}{4\cdot 4^{n-1}}=\underset{a}{\frac{1}{4}}\cdot\underset{r}{\left(\frac{2}{4}\right)}^{n-1}$

G.S.2. $\sum_{n=1}^{\infty}\frac{3^{n+1}}{4^n}$ $a=\frac{9}{4}$, $r=\frac{3}{4}$

$=\frac{\frac{9}{4}}{1-\frac{3}{4}}=9$ Hint: $\frac{3^{n+1}}{4^n}=\frac{3^2\cdot 3^{n-1}}{4\cdot 4^{n-1}}=\underset{a}{\frac{9}{4}}\cdot\underset{r}{\left(\frac{3}{4}\right)}^{n-1}$

Step 3: $\sum_{n=1}^{\infty}\frac{2^{n-1}-3^{n+1}}{4^n}=\frac{1}{2}-9=\boxed{-\frac{17}{2}}$ conv.

Question: Try to figure out what if we change the problem as $\sum_{n=0}^{\infty}\frac{2^{n-1}-3^{n+1}}{4^n}$

Hints for WW:

- Common tricks for G.S: $\frac{(-1)^n}{a^n}=\left(-\frac{1}{a}\right)^n$, $a^{n+1}=a\cdot a^n$, $a^n=a\cdot a^{n-1}$
  $a^n=\frac{a^{n+1}}{a}$, $a^{n-1}=\frac{a^n}{a}$
- ww4: $\left(\frac{1}{2}\right)^{\frac{n}{2}}=\left(\left(\frac{1}{2}\right)^{\frac{1}{2}}\right)^n=\left(\sqrt{\frac{1}{2}}\right)^n$, $r=\sqrt{\frac{1}{2}}=\frac{1}{\sqrt{2}}$.
- ww6: (Represent a repeating decimal as a Geometric Series)
  $0.\overline{6}=0.6666\cdots=0.6+0.06+0.006+\cdots=6\cdot\frac{1}{10}+6\cdot\frac{1}{10^2}+6\cdot\frac{1}{10^3}+\cdots+6\cdot\frac{1}{10^n}+\cdots$
- ww7, 8. Consider x as a fixed parameter. Find r in terms of x with $|r|<1$
- ww9. Div Test. Use l'Hopital to estimate $\lim_{n\to\infty}\left(1-\frac{1}{3n}\right)^n$ (exp type 1^∞)
- ww10. $\cos(7n\pi)=(-1)^n$. consider n is even and odd.

§11.3 The integral test and P-Series.

key points

- Series $\sum_{n=1}^{\infty} a_n \longleftrightarrow \int_1^{\infty} f(x)dx$ improper integral, conv/DIV simultaneously.
- p-Series: $\sum_{n=1}^{\infty} \frac{1}{n^p} \begin{cases} \text{conv.} & p>1 \\ \text{DIV} & p\le 1 \end{cases}$

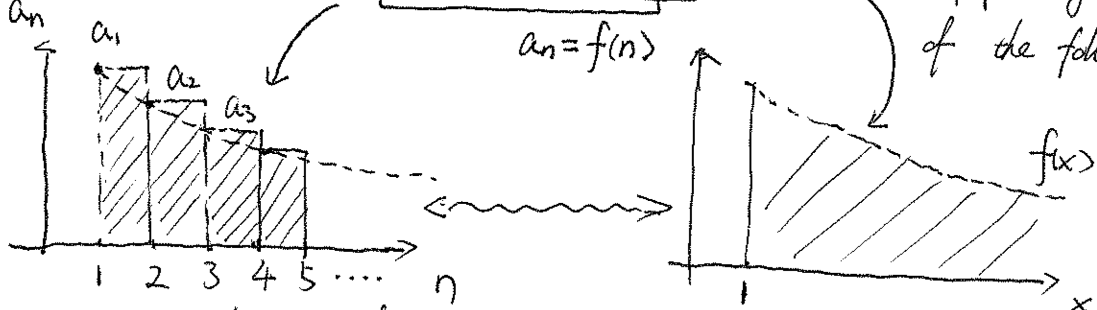
- Integral Test: Give $\sum_{n=1}^{\infty} a_n$. Rewrite a_n as $f(x)$. (Replace n by x in the formula of a_n).
  Assume that $f(x)$, i.e. ($f(n)=a_n$), is continuous, positive and DECREASING. We have
  ① If $\int_1^{+\infty} f(x)dx$ conv, then $\sum_{n=1}^{\infty} a_n$ conv. ② If $\int_1^{+\infty} f(x)dx$ DIV, then $\sum_{n=1}^{\infty} a_n$ DIV.

Motivation/Ideas of Integral Tests

$$\underbrace{\sum_{n=1}^{\infty}}\ \overbrace{a_n \sim \int_1^{+\infty} f(x)}\, dx.$$

$a_n = f(n)$

Improper integral represents the area of the following region.



Series represents the sum of the area of the rectangles: $a_1 \cdot 1 + a_2 \cdot 1 + a_3 \cdot 1 + \cdots$

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e.g.❶. Q: Can integral test be used to determine the conv/DIV of $\sum_{n=1}^{\infty} \sin(n)$?
(S16, 6pts). Answer: $\sin(n) \longleftrightarrow \sin x = f(x)$ is NOT positive and DECREASING. The integral test hypotheses ARE NOT met, so it cannot be applied.

Remark: $\lim_{n\to\infty} \sin(n)$ DOES NOT EXIST. So $\sum_{n=1}^{\infty} \sin(n)$ diverges due to DIV TEST.

- (Trivial) Example of Integral Test: $\sum_{n=1}^{\infty} \frac{1}{n}$, $a_n = \frac{1}{n}$, $f(x) = \frac{1}{x}$ continuous, positive and decreasing.

$$\int_1^{\infty} f(x)dx = \int_1^{+\infty} \frac{1}{x}dx = \lim_{t\to\infty} \int_1^t \frac{1}{x}dx = \lim_{t\to\infty} \ln|x|\Big|_1^t = \lim_{t\to\infty} \ln|t| = \ln(+\infty) = +\infty.$$

$\int_1^{\infty} f(x)dx$ diverges, & therefore, $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges.

e.g. 1. Determine whether the following series converges or diverges. State the test you use.

(s16, 10pts) $\sum_{n=2}^{\infty} \frac{5}{n\ln n}$. $\left(a_n = \frac{5}{n\ln n} \longleftrightarrow f(x) = \frac{5}{x\ln x}. \text{ satisfies } f(n) = a_n. \right)$

Solution: Since $f(x) = \frac{5}{x\ln x}$ is continuous, positive and decreasing for $n \geq 2$, the integral test can be applied.

$$\int_2^\infty \frac{5}{x\ln x}\,dx = \lim_{t\to\infty} \int_2^t \frac{5}{x\ln x}\,dx$$

Hint: $(\ln x, \frac{1}{x})$ pair, u-sub: $u = \ln x$, $du = \frac{1}{x}dx$

$$= \lim_{t\to\infty} \int_2^t \frac{5}{\ln x}\cdot\frac{1}{x}dx$$

$$= \lim_{t\to\infty} \int \frac{5}{u}\cdot du = \lim_{t\to\infty} 5\ln|u| = \lim_{t\to\infty} 5\ln|\ln x|\Big|_{x=2}^{x=t}$$

$$= \lim_{t\to\infty} 5\ln|\ln t| - 5\ln|\ln 2|$$

$$= \infty \quad (\text{Hint: } \ln\ln\infty = \ln\infty = \infty)$$

Therefore, $\int_2^\infty \frac{5}{x\ln x}dx$ diverges.

$\boxed{\text{So the series } \sum_{n=2}^{\infty} \frac{5}{n\ln n} \text{ diverges}}$. ※.

Remark: In general, similar argument works for $\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^p}$ with different parameters $p = 1, 2, 3, \ldots$

(ww 3, 4)

- Series of the form $\sum_{n=1}^{\infty} \frac{1}{n^p}$ is called p-series with parameter p (any number)

Conclusion of ConV/DIV: $\sum_{n=1}^{\infty} \frac{1}{n^p}$ is convergent if $p > 1$ and divergent if $p \leq 1$.

e.g. 2. Determine whether the following series is ConV or DIV.

- $\sum_{n=7}^{\infty} \frac{2}{n^3}$. p-series $p = 3 > 1$ ConV.
- $\sum_{n=1}^{\infty} -5n^{-\frac{1}{3}}$, $a_n = -5\cdot n^{-\frac{1}{3}} = -5\cdot\frac{1}{n^{\frac{1}{3}}}$, p-series, $p = \frac{1}{3} < 1$, DIV.
- $\sum_{n=2}^{\infty} \frac{4}{\sqrt{4n^e}}$. $a_n = \frac{4}{\sqrt{4n^e}} = \frac{4}{\sqrt{4}\cdot\sqrt{n^e}} = \frac{4}{2\cdot n^{\frac{e}{2}}}$, $p = \frac{e}{2} > 1$, ConV.
- $\sum_{n=1}^{\infty} \frac{3\sqrt{n}}{n}$, $a_n = \frac{3\sqrt{n}}{n} = \frac{3}{n^{\frac{1}{2}}}$, $p = \frac{1}{2} < 1$, DIV.

- Proof of conclusion on p-series via integral test. (Two typical p-values).

p-value	$p=2$ $(p>2)$	$p=\frac{1}{2}$ $(p<\frac{1}{2})$
p-series	$\sum_{n=1}^{\infty}\frac{1}{n^2}$	$\sum_{n=1}^{\infty}\frac{1}{n^{\frac{1}{2}}}$ (or $\sum_{n=1}^{\infty}\frac{1}{\sqrt{n}}$)
Improper integral	$\int_1^{+\infty}\frac{1}{x^2}\,dx$	$\int_1^{+\infty}\frac{1}{x^{\frac{1}{2}}}\,dx$.
Test for the integral	$=\lim_{t\to\infty}\int_1^t x^{-2}\,dx$. $=\lim_{t\to\infty} -x^{-1}\Big\vert_1^t$ $=\lim_{t\to\infty} -\frac{1}{t}+1=\boxed{1}$	$=\lim_{t\to\infty}\int_1^t x^{-\frac{1}{2}}\,dx$. Hint: $\int x^{-\frac{1}{2}}=\frac{1}{-\frac{1}{2}+1}x^{-\frac{1}{2}+1}=2\cdot x^{\frac{1}{2}}$ $=\lim_{t\to\infty} 2\sqrt{x}\Big\vert_1^t$ $=\lim_{t\to\infty} 2\sqrt{t}-2\sqrt{1}=\boxed{+\infty}$
Conclusion:	$\int_1^{+\infty}\frac{1}{x^2}dx$ conv $\Rightarrow \sum_{n=1}^{\infty}\frac{1}{n^2}$ conv.	$\int_1^{+\infty}\frac{1}{x^{\frac{1}{2}}}dx$ DIV $\Rightarrow \sum_{n=1}^{\infty}\frac{1}{\sqrt{n}}$ DIV.

More remarks:

- p-series VS Geometric Series:

$\sum_{n=1}^{\infty}\frac{1}{n^p}$ $\qquad \sum_{n=1}^{\infty}a\cdot r^{n-1}=\sum_{n=0}^{\infty}a\cdot r^n$

p, r / Condition	p-Series.	G.S.
Convergent	$p>1$	$\lvert r\rvert<1$
Divergent.	$p\leq 1$	$\lvert r\rvert\geq 1$.

G.S. has exact value which can be computed via $\sum_{n=0}^{\infty}a\cdot r^n=\frac{a}{1-r}$, $\lvert r\rvert<1$

p-Series ('s exact value) can not be evaluated. The value of improper integral is NOT equal to the sum of p-Series.

$\sum_{n=1}^{\infty}\frac{1}{n^{\frac{2}{3}}}$ VS $\sum_{n=1}^{\infty}\left(\frac{2}{3}\right)^n$,

$p=\frac{2}{3}<1$, $\sum_{n=1}^{\infty}\frac{1}{n^{\frac{2}{3}}}$ DIV.

$r=\frac{2}{3}<1$, $\sum_{n=1}^{\infty}\left(\frac{2}{3}\right)^n$ conv. $\left(=\frac{\frac{2}{3}}{1-\frac{2}{3}}=2\right)$

$\left(a=\frac{2}{3},\ r=\frac{2}{3}.\right)$

- In the following sections, we will consider series which looks similar either to p-series (such as $\sum_{n=1}^{\infty}\frac{1}{(2n+1)^3}$, $\sum_{n=2}^{\infty}\frac{n^2-1}{3n^3+1}$) or a geometric series (such as $\sum\frac{1}{(3^n+7)}$), or a combination of these two (such as $\sum\frac{n^2}{7^n}$).