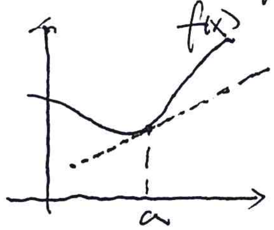
§11.10 Taylor and Maclaurin Series.

Motivation: Tangent line and Linear Approximation of $f(x)$ at $x=a$.



$$L(x) = f(a) + f'(a)\cdot(x-a) \approx f(x)$$

We use a linear function $L(x)$ to approximate $f(x)$ near $x=a$.

Question: How to get more accurate approximation for $f(x)$?

Answer: Instead of using linear function, we can consider higher degree polynomials.

Definition: The Taylor Series of a function $f(x)$ at $x=a$ is the following Power Series:

$$☆ \quad \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}\cdot(x-a)^n = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + \frac{f^{(4)}(a)}{4!}(x-a)^4 + \cdots\cdots$$

(Formula Sheet) The Maclaurin Series of $f(x)$ is the Taylor Series at $x=0$ (i.e., $a=0$)

$$\bullet \quad \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!}\cdot x^n = f(0) + f'(0)x + \frac{f''(0)}{2}\cdot x^2 + \frac{f'''(0)}{3!}x^3 + \frac{f^{(4)}(0)}{4!}\cdot x^4 + \cdots\cdots$$

The truncated sum at n-th degree is called nth degree Taylor Polynomial

$$☆ \quad T_n(x) = \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!}(x-a)^k = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots\cdots + \frac{f^{(n-1)}(a)}{(n-1)!}(x-a)^{n-1} + \frac{f^{(n)}(a)}{n!}\cdot(x-a)^n$$

Remarks:

① $f^{(n)}(a)$ means the nth order derivative of $f(x)$ evaluated at $x=a$. For example, $f^{(4)}(2) = f''''(2)$

② Taylor/Maclaurin Series are Power Series centered at $x=a$ with coefficients $c_n = \frac{f^{(n)}(a)}{n!}$

③ Five frequently used Power Series (Maclaurin Series) are given in the formula sheet.

$$e^x = \sum_{n=0}^{\infty} \frac{1}{n!}\cdot x^n \quad ; \quad \ln(1+x) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} x^n \quad ; \quad \frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$$

$$\sin x = \sum_{n=0}^{\infty} (-1)^n \cdot \frac{x^{2n+1}}{(2n+1)!} \quad ; \quad \cos x = \sum_{n=0}^{\infty} (-1)^n \cdot \frac{x^{2n}}{(2n)!}$$

eg 1. Find the second-degree Taylor polynomial generated by (s16, MC) $f(x)=\frac{1}{x}$ about the pointe $x=3$.

Remark: Apply the formula of $T_n(x)$ at $a=3$. The key step is the $\boxed{\text{DERIVATIVE TABLE}}$

Solution: Derivative table of $f(x)=\frac{1}{x}$ at $a=3$ up to degree $n=2$.

$f \setminus n$	$n=0$	$n=1$	$n=2$
$f^{(n)}(x)$	$f^{(0)}(x)=f(x)=\frac{1}{x}$	$f'(x)=\left(\frac{1}{x}\right)'=\frac{-1}{x^2}$	$f''(x)=\left(-\frac{1}{x^2}\right)'=(-1)\cdot(-2)\cdot\frac{1}{x^3}=\frac{2}{x^3}$
$f^{(n)}(3)$	$f(3)=\frac{1}{3}$	$f'(3)=\frac{-1}{3^2}=-\frac{1}{9}$	$f''(3)=\frac{2}{3^3}=\frac{2}{27}$

Therefore, $T_2(x)\overset{a=3}{=}f(3)+f'(3)\cdot(x-3)+\frac{f''(3)}{2!}\cdot(x-3)^2=\boxed{\frac{1}{3}-\frac{1}{9}(x-3)+\frac{2}{27}\cdot\frac{(x-3)^2}{2!}}$

eg 2. Find the 3rd degree Maclaurin polynomial of $f(x)=e^{-5x}$

Solution: Direct application of the formula for $e^x=\sum_{n=0}^{\infty}\frac{x^n}{n!}$

$$e^{-5x}=\sum_{n=0}^{\infty}\frac{(-5x)^n}{n!}=\sum_{n=0}^{\infty}\frac{(-5)^n}{n!}\cdot x^n=1-5x+\frac{(-5x)^2}{2!}+\frac{(-5x)^3}{3!}+\cdots$$

Therefore, $\boxed{T_3(x)=1-5x+\frac{25}{2}\cdot x^2-\frac{5^3}{3!}\cdot x^3}$, and the nth term is $\boxed{\frac{(-5)^n}{n!}\cdot x^n}$.

Remark: These formulas only apply to Maclaurin Series. For Taylor Series, you still need to set the Derivative Table.

eg 2'. Find the 3rd degree Taylor polynomial of $f(x)=e^{-5x}$ at $a=-1$.

$f \setminus n$	$n=0$	$n=1$	$n=2$	$n=3$
$f^{(n)}(x)$	e^{-5x}	$(-5)e^{-5x}$	$(-5)^2\cdot e^{-5x}$	$(-5)^3\cdot e^{-5x}$
$f^{(n)}(-1)$	e^5	$-5\cdot e^5$	$25\cdot e^5$	$-125\, e^5$

Therefore, $\boxed{T_3(x)=e^5-5e^5\cdot(x+1)+\frac{25\cdot e^5}{2!}\cdot(x+1)^2-\frac{125\cdot e^5}{3!}\cdot(x+1)^3}$

Remark: The Derivative Table method also works for eg 2. Maclaurin Series with $a=0$.

eg.3. Find the Taylor Series of $f(x) = x^3 + x - 5$ at $x=2$

s|n: $f^{(n)}(x)$ $f^{(n)}(2)$

$n=0$ $f(x) = x^3+x-5$ $f(2)=5$

$n=1$ $f'(x)=3x^2+1$ $f'(2)=13$

$n=2$ $f''(x)=6x$ $f''(2)=12$

$n=3$ $f'''(x)=6$ $f'''(2)=6$

$n\geq 4$ $f^{(n)}(x)=0$ $f^{(n)}(2)=0$

Derivative Table implies:

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(2)}{n!} \cdot (x-2)^n$$

$$= \frac{5}{0!}(x-2)^0 + \frac{13}{1!}(x-2)^1 + \frac{12}{2!}(x-2)^2 + \frac{6}{3!}(x-2)^3 + 0$$

$$= \boxed{5 + 13\cdot(x-2) + 6\cdot(x-2)^2 + (x-2)^3}$$

Rmk: The Taylor Series of a polynomial is still a polynomial with the same order.

eg4. Find the 3rd degree Taylor polynomial of $f(x)=(x+1)^{\frac{3}{2}}$ centered at $x=3$.
(S17).

s|n: $f(x)=(x+1)^{\frac{3}{2}}$ $f(3)=4^{\frac{3}{2}}=(\sqrt{4})^3=8$

$f'(x)=\frac{3}{2}(x+1)^{\frac{1}{2}}$ $f'(3)=\frac{3}{2}\cdot 4^{\frac{1}{2}}=\frac{3}{2}\sqrt{4}=3$

$f''(x)=\frac{1}{2}\cdot\frac{3}{2}\cdot(x+1)^{-\frac{1}{2}}$ $f''(3)=\frac{1}{2}\cdot\frac{3}{2}4^{-\frac{1}{2}}=\frac{1}{2}\cdot\frac{3}{2}\cdot\frac{1}{\sqrt{4}}=\frac{3}{8}$

$f'''(x)=(-\frac{1}{2})\cdot\frac{1}{2}\cdot\frac{3}{2}(x+1)^{-\frac{3}{2}}$ $f'''(3)=-\frac{1}{2}\frac{1}{2}\cdot\frac{3}{2}\cdot 4^{-\frac{3}{2}}=-\frac{1}{2}\cdot\frac{1}{2}\cdot\frac{3}{2}\cdot\frac{1}{(\sqrt{4})^3}=-\frac{3}{64}$

The 3rd degree Taylor Polynomial is:

$$T_3(x)=f(3)+f'(3)(x-3)+\frac{f''(3)}{2}\cdot(x-3)^2+\frac{f'''(3)}{3!}\cdot(x-3)^3$$

$$=\boxed{8+3\cdot(x-3)+\frac{\frac{3}{8}}{2}(x-3)^2+\frac{-\frac{3}{64}}{3!}\cdot(x-3)^3}$$

☆ eg.5. Let $\sin x = \sum_{n=0}^{\infty} C_n\cdot(x-\frac{\pi}{6})^n$ be the Taylor series for $\sin x$ at $a=\frac{\pi}{6}$. Find C_2.
(S17)

s|n: $C_2 = \frac{f''(\frac{\pi}{6})}{2!}$ where $f(x)=\sin x$. $(\sin x)'=\cos x$, $(\sin x)''=(\cos x)'=-\sin x$.

$$\Rightarrow C_2 = \frac{-\sin\frac{\pi}{6}}{2!} = \frac{-\frac{1}{2}}{2} = \boxed{-\frac{1}{4}}$$

☆☆☆ eg.6 Evaluate the indefinite integral $\int \frac{\sin x}{x}dx$ as a power series
(s17)

sln: Formula sheet: $\sin x = \sum_{n=0}^{\infty}(-1)^n \cdot \frac{x^{2n+1}}{(2n+1)!} \Rightarrow \frac{\sin x}{x} = \frac{1}{x}\cdot\sum_{n=0}^{\infty}(-1)^n\cdot\frac{x^{2n+1}}{(2n+1)!}$

$$= \sum_{n=0}^{\infty}\frac{1}{x}\cdot(-1)^n\cdot\frac{x^{2n+1}}{(2n+1)!}$$

$$= \sum_{n=0}^{\infty}(-1)^n\cdot\frac{x^{2n}}{(2n+1)!}$$

$$\int \frac{\sin x}{x}dx = \int \sum_{n=0}^{\infty}(-1)^n\cdot\frac{x^{2n}}{(2n+1)!}\cdot dx$$

$$= \sum_{n=0}^{\infty}(-1)^n\cdot\frac{\int x^{2n}dx}{(2n+1)!}$$

$$= \sum_{n=0}^{\infty}(-1)^n\cdot\frac{\frac{1}{2n+1}\cdot x^{2n+1}}{(2n+1)!} + C = \boxed{\sum_{n=0}^{\infty}(-1)^n\cdot\frac{x^{2n+1}}{(2n+1)\cdot(2n+1)!} + C}$$

• eg7 Find the first 4 nonzero terms of the Taylor series at $x=0$ for
(f16) $f(x) = 4x\cdot\cos x - 3 + 2x^3$

sln: Formula sheet: $\cos x = \sum_{n=0}^{\infty}(-1)^n\cdot\frac{x^{2n}}{(2n)!} = 1 - \frac{x^2}{2} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots$

$$\Rightarrow f(x) = 4x\cdot\left(1 - \frac{x^2}{2} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots\right) - 3 + 2x^3$$

$$= 4x - 2x^3 + \frac{4x^5}{4!} - \frac{4x^7}{6!} + \cdots - 3 + 2x^3 = \boxed{-3 + 4x + 0 + \frac{4x^5}{4!} - \frac{4x^7}{6!}} + \cdots$$

Rmk: The derivative table method also works.

• eg8. Find the Maclaurin Series for $f(x) = x\cdot\cos\sqrt{x}$.
(f15).

sln: $f(x) = x\cdot\sum_{n=0}^{\infty}(-1)^n\frac{(\sqrt{x})^{2n}}{(2n)!}$. since $\cos ▨ = \sum_{n=0}^{\infty}(-1)^n\cdot\frac{▨^{2n}}{(2n)!}$

$$= x\cdot\sum_{n=0}^{\infty}(-1)^n\cdot\frac{x^n}{(2n)!}$$. since $(\sqrt{x})^{2n} = ((\sqrt{x})^2)^n = x^n$

$$\boxed{= \sum_{n=0}^{\infty}(-1)^n\cdot\frac{x^{n+1}}{(2n)!}}$$

Q: what's the power series of $\int x\cdot\cos\sqrt{x}\,dx$?

$$\cancel{\int} = \int \sum_{n=0}^{\infty}(-1)^n\cdot\frac{x^{n+1}}{(2n)!}dx = \sum_{n=0}^{\infty}(-1)^n\cdot\frac{\int x^{n+1}dx}{(2n)!} = \boxed{\sum_{n=0}^{\infty}(-1)^n\cdot\frac{1}{(2n)!}\cdot\frac{1}{n+2}x^{n+2} + C}$$

§11.10/11. Application of Taylor/Maclaurin Series.

- The Maclaurin Series of $f(x)=(1+x)^k$ is also called Binomial Series

Actually, direct computation shows $(1+x)^k = 1 + kx + \frac{k\cdot(k-1)}{2!}\cdot x^2 + \frac{k(k-1)(k-2)}{3!}\cdot x^3 + \cdots$

(~~the~~ formula is NOT required).

eg1. Find the 4th term of the binomial series for $f(x)=\frac{1}{\sqrt[5]{1+x^2}}$
(ww3)

sln: $f(x) = (1+x^2)^{-\frac{1}{5}}$, $k=-\frac{1}{5}$

The coefficient of the 4th term is $\frac{k\cdot(k-1)(k-2)}{3!} = \frac{(-\frac{1}{5})\cdot(-\frac{1}{5}-1)(-\frac{1}{5}-2)}{3!} = -\frac{77}{3!\cdot 5^3}$

The 4th term is $-\frac{77}{3!\cdot 5^3}\cdot(x^2)^3 = \boxed{-\frac{77}{3!\cdot 5^3}\cdot x^6}$.

- The Taylor and Maclaurin series can be used for integral.

eg2 Find the first 4 non-zero terms of the Taylor Series of
(WW5)

$$F(x) = \int_0^x 3t^2\cdot\cos t\cdot dt \quad \text{at } x=0.$$

sln: $3t^2\cdot\cos t = 3t^2\cdot(1-\frac{t^2}{2}+\frac{t^4}{4!}-\frac{t^6}{6!}+\cdots) = 3t^2 - \frac{3t^4}{2} + \frac{3t^6}{4!} - \frac{3t^8}{6!} + \cdots$

$$\Rightarrow F(x) = \int_0^x 3t^2\cos t\cdot dt = \int_0^x 3t^2 - \frac{3t^4}{2} + \frac{3t^6}{4!} - \frac{3t^8}{6!} + \cdots dt$$

$$= 3\cdot\frac{1}{3}t^3 - \frac{3}{2}\cdot\frac{1}{5}t^5 + \frac{3}{4!}\cdot\frac{1}{7}t^7 - \frac{3}{6!}\cdot\frac{1}{9}t^9 + \cdots \Big|_0^x$$

$$= \boxed{3\cdot\frac{1}{3}x^3 - \frac{3}{2}\cdot\frac{1}{5}\cdot x^5 + \frac{3}{4!}\cdot\frac{1}{7}x^7 - \frac{3}{6!}\cdot\frac{1}{9}\cdot x^9} + \cdots$$

☆ • The power series expansion can be used to evaluate the limit for $x\to 0$.

eg3. $\lim_{x\to 0}\frac{\cos(x^2)-1}{x^4} \overset{L'H}{=} \lim_{x\to 0}\frac{-\sin(x^2)\cdot 2x}{4x^3} = \lim_{x\to 0}\frac{-\sin(x^2)}{2\cdot x^2} = \boxed{-\frac{1}{2}}$

If we see $\cos x^2$ is power series, we have $\cos x^2 = 1 - \frac{1}{2}x^4 + \frac{1}{4}\cdot x^8 + \cdots$

$$\Rightarrow \frac{\cos(x^2)-1}{x^4} = \frac{(1-\frac{1}{2}x^4+\frac{1}{4}x^8+\cdots)-1}{x^4} = \frac{-\frac{1}{2}x^4+\frac{1}{4}x^8+\cdots}{x^4} = \boxed{-\frac{1}{2}}$$

eg.4. Evaluate the following limit by power series: $\lim\limits_{x\to 0} \dfrac{\frac{1}{1+3x^2} - 1 + 3x^2 - 9x^4}{5x^6}$
(ww6)

s/n: Express $\frac{1}{1+3x^2}$ as $\sum\limits_{n=0}^{\infty}(-3x^2)^n = 1 - 3x^2 + (3x^2)^2 - (3x^2)^3 + \cdots$

$$= 1 - 3x^2 + 9x^4 - 27\cdot x^6 + \cdots$$

$$\Rightarrow \lim_{x\to 0} \frac{\frac{1}{1+3x^2} - 1 + 3x^2 - 9x^4}{5\cdot x^6} = \lim_{x\to 0} \frac{-27\cdot x^6 + \cdots}{5\cdot x^6} = \boxed{-\frac{27}{5}}$$

☆ For a Taylor Series

$$\sum_{k=0}^{\infty} \frac{f^{(k)}(a)}{k!}(x-a)^k = \underbrace{\sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!}(x-a)^k}_{T_n(x)} + \underbrace{\sum_{k=n+1}^{\infty} \frac{f^{(k)}(a)}{k!}(x-a)^k}_{R_n(x)}$$

We call $R_n(x) = \sum\limits_{k=n+1}^{\infty} \frac{f^{(k)}(a)}{k!}(x-a)^k$ the REMAINDER of the Taylor Series.

We have the following Taylor's Inequality (in formula sheet). If $|f^{(n+1)}(x)| \le M$ for $|x-a| \le d$, then the remainder $R_n(x)$ of the Taylor series satisfies

$$|R_n(x)| \le \frac{M}{(n+1)!}|x-a|^{n+1} \quad \text{for } |x-a| \le d.$$

eg.5. We want use the 3rd degree Taylor Polynomial at $x=0$ to estimate e^x on $[-1,1]$.
Find $T_3(x)$, $R_3(x)$. What's the error for using $T_3(0.5)$ to estimate $e^{0.5}$?

s/n: $e^x = 1 + x + \frac{1}{2}x^2 + \frac{1}{3!}x^3 + \frac{1}{4!}x^4 + \cdots$, $T_3(x) = 1 + x + \frac{1}{2}x^2 + \frac{1}{3!}x^3$

$R_3(x) = e^x - T_3(x) = \frac{1}{4!}x^4 + \frac{1}{5!}x^5 + \cdots$

$e^{0.5} \sim T_3(0.5) = 1 + 0.5 + \frac{1}{2}\cdot 0.5^2 + \frac{1}{3!}\cdot 0.5^3$

The error between $e^{0.5}$ and $T_3(0.5)$ is $R_3(0.5)$. According to Taylor's Inequality
$|R_3(0.5)| \le \frac{M}{(3+1)!}|0.5 - a|^{3+1}$. M is the maximal value of $f^{(4)}(x) = e^x$ on $[-1,1]$
i.e. $M = e$, a is the center of the series $a=0$, d is the radius $d=1$

$|R_3(0.5)| \le \frac{e}{4!}(0.5)^4$ (We can get some numerical estimates if we assume $e \le 2.8$)