Introduction. Review for Definite integral / Area.

- Goals: Problem solving techniques / Math way of Thinking

- (Definite) Integral:

Motivation: How to measure the area of (half) unit disk?

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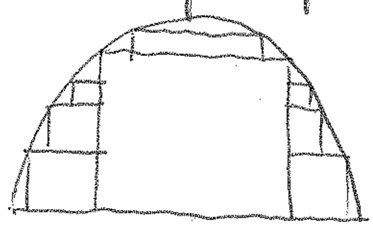
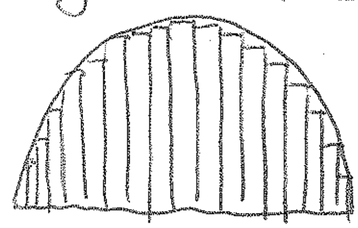
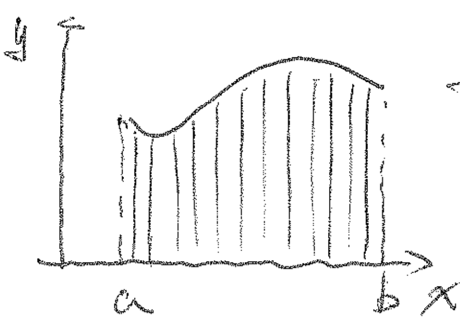


Approximation by rectangles



Order / Arrangement.

In general

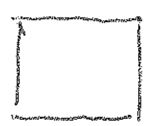


$$A = \int_a^b f(x)\,dx.$$

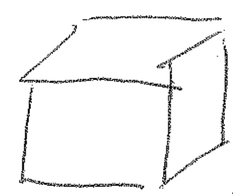
- 

point.

Line (length)



Plane region (Area)



Solid (Volume)

Differentiation $\longleftrightarrow$ Integration.

Acceleration Velocity Displacement (Distance)

Force. Work.

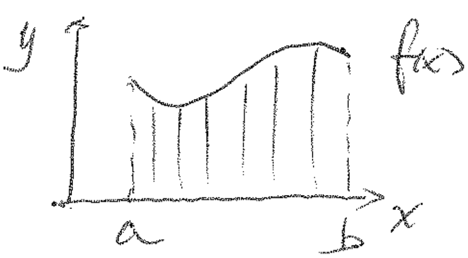
- Set up the integral (Chapt 5, 6)

- Evaluate the integral (Chapt 7)

(Fundamental Theory of Calculus)

$\int_a^b f(x)\,dx = F(b) - F(a)$, where $F(x)$ is an Anti-Derivative of $f(x)$.

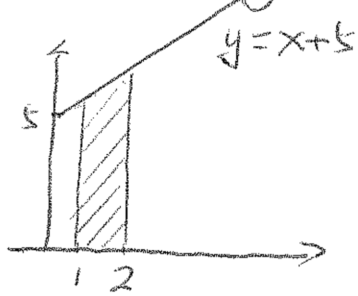
Area Between Curves (Review of §5.1)



$\longleftrightarrow$ Area $= \int_a^b f(x)\,dx$.

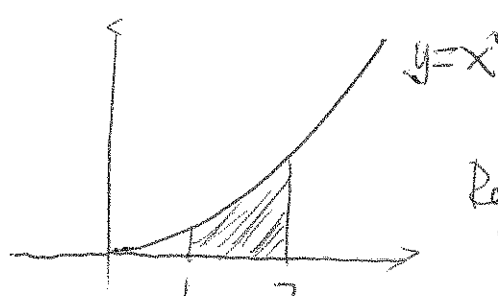
dx suggests "sum up along x-direction".

e.g. $\int_1^2 x+5\,dx \longleftrightarrow \int$.

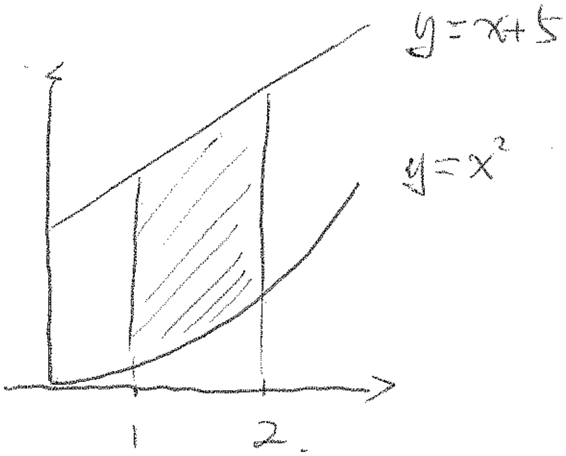


Region I

$\int_1^2 x^2\,dx \longleftrightarrow$



Region II

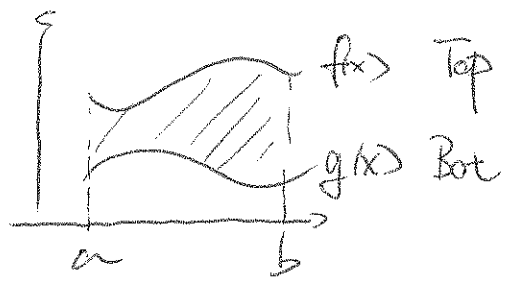


Area = Region I − Region II

$$= \int_1^2 x+5 \, dx - \int_1^2 x^2 \, dx$$

$$= \int_1^2 x+5 - x^2 \, dx.$$

- In general,



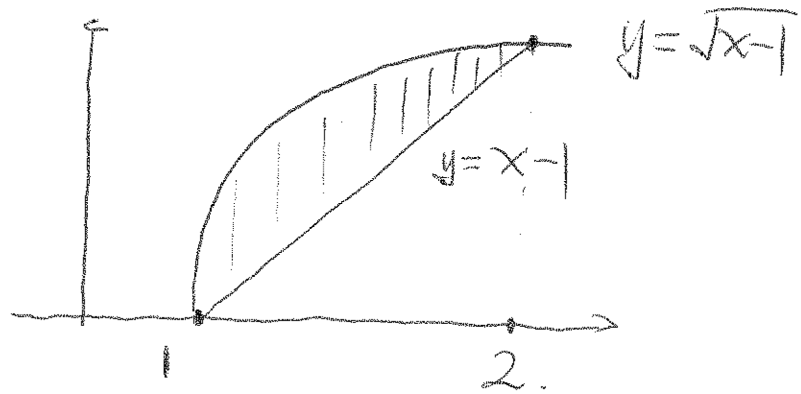
$$\text{Area} = \int_a^b f(x) - g(x) \, dx.$$

- e.g. Sketch the region enclosed by the given curves and set up the integral for the area.

$y = \sqrt{x-1}$, $x - y = 1$.

$\Downarrow$

$(y = x-1)$



$$\text{Area} = \int_1^2 \sqrt{x-1} - (x-1) \, dx.$$

§5.2 Volumes

Key points:

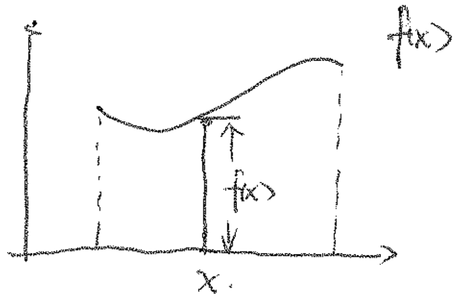
① $V = \pi \int_a^b \text{Radius}^2 \, dx$ Disk

② $V = \pi \int_a^b \text{Outer Radius}^2 - \text{Inner Radius}^2 \, dx$. Washer

} Solid of Revolution

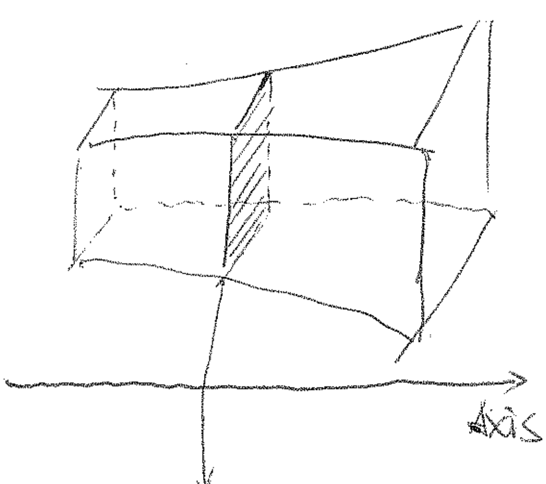
③ $V = \int_a^b \text{Side length}^2 \, dx$. Pyramid with Square Cross-Section.

☆☆ ④ $V = \int_a^b A(x) \cdot dx$. General Cross Section with Area $A(x)$



$$\text{Area} = \int_a^b \underbrace{f(x)}_{\text{length(Height) of "Cross Section"}} \, dx = \int_a^b \text{length} \, dx.$$

⇓ Generalization

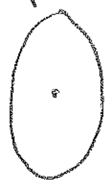


$$\text{Volume} = \int_a^b \underbrace{A(x)}_{\text{Area of the Cross-Section}} \, dx$$

dx shows the direction that cross-section is moving along.

Cross Section (Perpendicular to axis)

Two Types of Cross-Sections.

- Disk. 

(Washer = two disks 

Rotating a plane curve/region.

- Square Cross-Section

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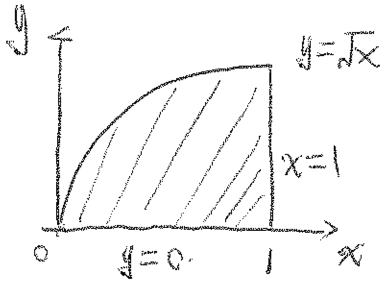
e.g.1 (Disk) ① Sketch the region bounded by $y=\sqrt{x}$, $y=0$, $x=1$.

② Rotate the region w.r.t. x-axis, describe the solid.

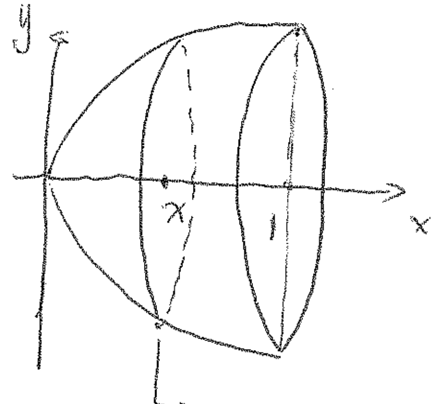
③ Set up an integral for the volume of the rotating solid.

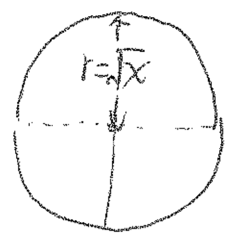
④ Find the volume.

①



②



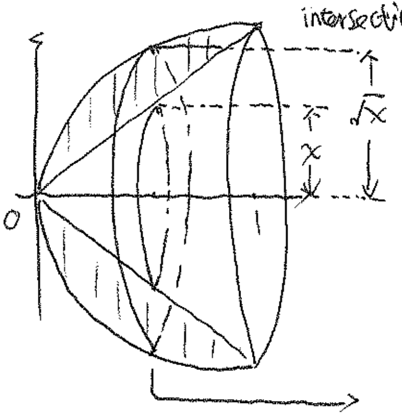


Cross Section is a DISK with radius $r=\sqrt{x}$. Area $=\pi\cdot r^2$
$=\pi(\sqrt{x})^2$

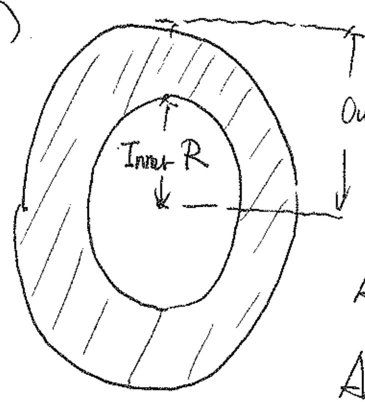
③ $$\boxed{V=\int_0^1 \pi\cdot(\sqrt{x})^2\,dx}$$

④ $$V=\int_0^1 \pi\cdot x\,dx = \pi\cdot\frac{1}{2}x^2\Big|_0^1 = \pi\cdot\frac{1}{2}\cdot 1^2 - \pi\cdot\frac{1}{2}\cdot 0^2$$
$$=\boxed{\frac{1}{2}\cdot\pi}$$

eg.2 (Washer) Region enclosed by $y=\sqrt{x}$, $y=x$, rotated w.r.t. x-axis.



Project the cross-section on the plane



Outer Radius $R_{outer}=\sqrt{x}$.

$R_{inner}=x$

Area of Cross-Section:

$$A(x)=\pi\cdot R_{outer}^2-\pi\cdot R_{inner}^2$$
$$=\pi\cdot(\sqrt{x})^2-\pi\cdot x^2$$

$$\text{Volume}=\int_0^1 A(x)\cdot dx=\int_0^1[\pi\cdot\sqrt{x}^2-\pi\cdot x^2]\cdot dx.$$ (Set-up integral only).

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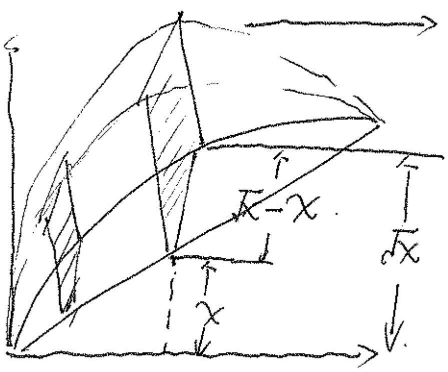
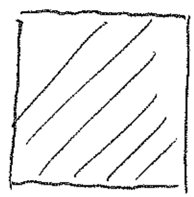
eg.3. (Pyramid) Let the region be the same as in eg.2. The cross section is a square perpendicular to x-axis with side in X-Y plane.

① Describe the solid. ② Find (Set up) the volume.



Cross-Section is a Square with side $L=\sqrt{x}-x$.



$L=\sqrt{x}-x$.

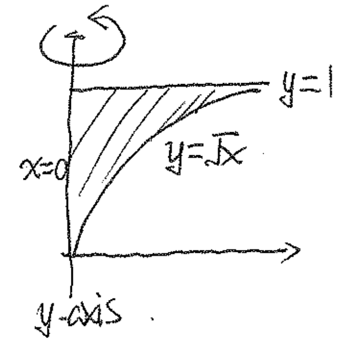
Area of C.S: $A(x)=L^2=(\sqrt{x}-x)^2$

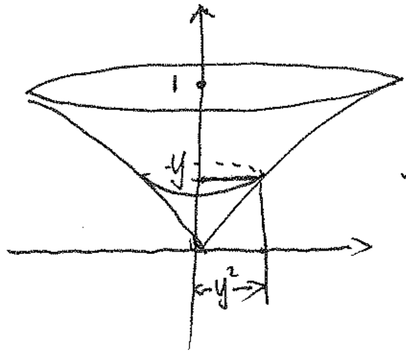
$$\text{Volume}=\int_0^1 A(x)\cdot dx=\boxed{\int_0^1(\sqrt{x}-x)^2\cdot dx}$$

☆ Rotating about y-axis.

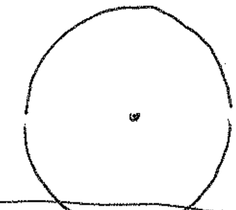
e.g. 4. Region enclosed by $y=\sqrt{x}$, $x=0$, $y=1$

Rotate the region about y-axis. Find the volume of the solid.





Cross. Section →



$\boxed{\text{Radius} = x = y^2}$

$A(y) = \pi \cdot \text{Radius}^2 = \pi \cdot (y^2)^2 = \pi y^4$

Note: The cross-section is moving along y-direction. The integral is set up with respect to $\boxed{y\text{ - variable}}$.

$$V = \int_0^1 A(y) \cdot dy$$

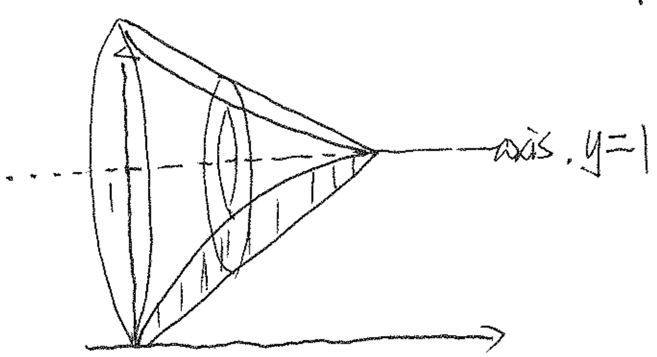
$$= \boxed{\int_0^1 \pi \cdot y^4 \cdot dy}.$$

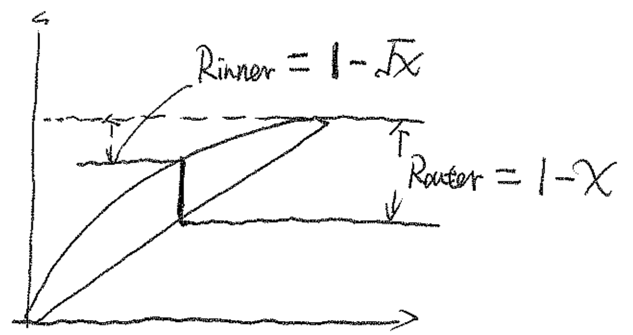
☆☆. Rotating about OTHER axis.

eg. 5. Region: $y=\sqrt{x}$, $y=x$, rotated about $y=1$.

Note: $y=1$ is a HORIZONTAL axis (parallel to x axis)

The integral is set up with r.t. $\boxed{x}$.



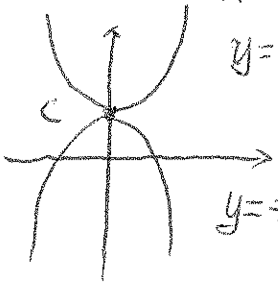


Area of Cross-Section: $A(x) = \pi \cdot R_{outer}^2 - \pi \cdot R_{inner}^2$

$$= \pi \cdot (1-x)^2 - \pi \cdot (1-\sqrt{x})^2$$

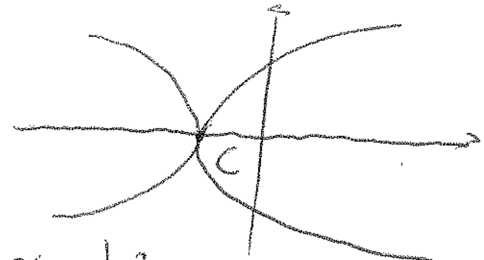
$$\text{Volume} = \int_0^1 A(x) \cdot dx = \boxed{\int_0^1 \pi \cdot (1-x)^2 - \pi \cdot (1-\sqrt{x})^2 \cdot dx}.$$

Appendix. graph of frequently used functions.

☆ 

$y=x^2+C$

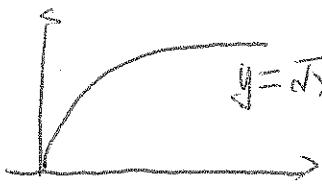
$y=-x^2+C$

☆ 

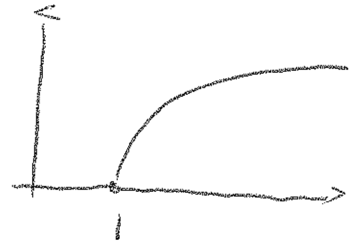
$x=-\frac{1}{4}y^2+C$ $x=\frac{1}{4}y^2+C$

(horizontal and vetical parabola)

☆☆ half parabola $y=\sqrt{ax+b}$

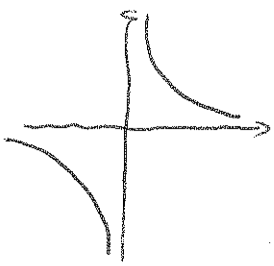
e.g. 

$y=\sqrt{x}$.

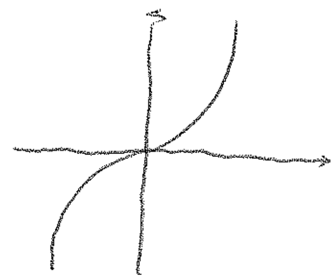


$y=\sqrt{x-1}$

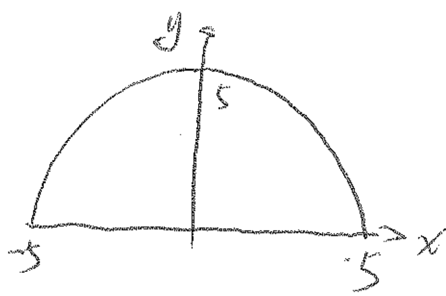
☆ $y=\frac{1}{x}$



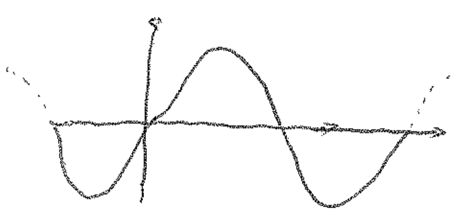
$y=x^3$



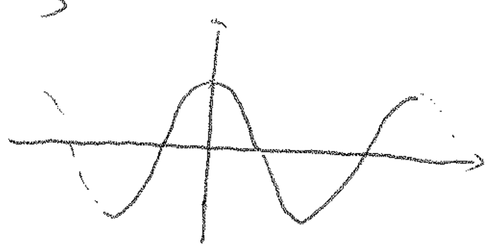
☆ $y=\sqrt{25-x^2}$ (half circle)



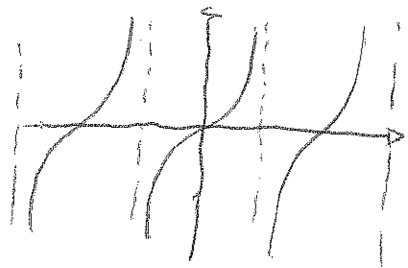
☆ $y=\sin x$



$y=\cos x$



☆☆ $y=\tan x$.



☆☆☆ (chapter 6) $y=e^x$, $y=\ln x$.

§4.5. Substitution Method

Key points: ① Differential Notation: If $u=g(x)$, then $du=g'(x)\cdot dx$.

② u-Sub: $\int f(\underbrace{g(x)}_{u})\cdot \underbrace{g'(x)\,dx}_{du} \ \xrightarrow[du=g'(x)dx]{u=g(x)} \ \int f(u)\cdot du$

• Goal of u-Substitution Method: By changing the variable x into u, we convert the integral into one of those five basic integrals which we can deal with.

• Basic integrals: $\int x^n dx = \frac{1}{n+1}x^{n+1}+C$, $\int \sin x\,dx = -\cos x + C$, $\int \cos x\,dx = \sin x + C$

$\int \sec^2 x\,dx = \tan x + C$, $\int \sec x\cdot\tan x\,dx = \sec x + C$.

① Differential Notation and Composition of functions.

eg.1. $u=x^2+1$, $\frac{du}{dx}=(x^2+1)'=2x \Rightarrow \boxed{du = 2x\cdot dx.}$

eg2. If $f(x)=\sqrt{x}$, ~~then~~ $u=x^2+1$, then $f(u)=\sqrt{u}=\sqrt{x^2+1}$

② u-Sub method for indefinite integral

eg.3. Evaluate $\int \sqrt{u}\cdot du \overset{n=\frac{1}{2}}{=\!=} \frac{1}{\frac{1}{2}+1}u^{\frac{1}{2}+1}+C = \boxed{\frac{2}{3}u^{\frac{3}{2}}+C}$

eg.4. $\int \sqrt{x^2+1}\cdot 2x\cdot dx$.

$=\int \sqrt{u}\cdot du$.

$=\frac{2}{3}u^{\frac{3}{2}}+C$

$=\frac{2}{3}(x^2+1)^{\frac{3}{2}}+C$.

Set $u=x^2+1$, According to eg1. $du=2x\cdot dx$.

Plug in u and du. By substituting u and du the integral of x turns into an integral of u with simpler form, which we evaluate in eg3.

Replace u by x^2+1 in the last step.

• The key of u-sub is to find the right substitution u.

More examples:

eg.5 Evaluate $\int x\cdot(2x^2+1)^3\cdot dx$. Hint: $u=2x^2+1$ is the ugly part.
(14 Final)

$u=2x^2+1$, $du=4x\cdot dx$, $\Rightarrow \boxed{x\cdot dx=\frac{1}{4}\cdot du}$

$$=\int (2x^2+1)^3\cdot x\cdot dx$$

$$=\int u^3\cdot\frac{1}{4}\cdot du=\frac{1}{4}\cdot\frac{1}{3+1}\cdot u^{3+1}+C$$

$$=\frac{1}{16}\cdot u^4+C=\boxed{\frac{1}{16}(2x^2+1)^4+C}$$

e.g.6 Evaluate $\int\frac{\sec(\frac{x}{2})\cdot\tan(\frac{x}{2})}{\sqrt{\sec(\frac{x}{2})}}\cdot dx$. Hint: $u=\sec(\frac{x}{2})$ is the ugly part.
(14 Final)

$u=\sec(\frac{x}{2})$, $du=\sec(\frac{x}{2})\tan(\frac{x}{2})\cdot\frac{1}{2}\cdot dx$. chain rule for $(\sec\blacksquare)'=\sec\blacksquare\cdot\tan\blacksquare$

$$\Rightarrow 2du=\sec(\tfrac{x}{2})\cdot\tan(\tfrac{x}{2})\cdot dx$$

$$=\int\frac{2du}{\sqrt{u}}=\int 2\cdot u^{-\frac{1}{2}}\,du$$

$$=2\cdot\frac{1}{-\frac{1}{2}+1}\cdot u^{-\frac{1}{2}+1}+C$$

$$=2\cdot 2\cdot u^{\frac{1}{2}}+C \xlongequal{\text{back to } x} \boxed{4\left(\sec(\tfrac{x}{2})\right)^{\frac{1}{2}}+C}$$

• Linear substitution and more general form of the five basic integrals.

☆ $\int(ax+b)^n=\frac{1}{n+1}\cdot x^{n+1}+C$, $\int\sin(ax+b)dx=\frac{-1}{a}\cos(ax+b)+C$, $\int\cos(ax+b)dx=\frac{1}{a}\sin(ax+b)+C$

$\int\tan(ax+b)\cdot\sec(ax+b)\cdot dx=\frac{1}{a}\sec(ax+b)+C$, $\int\sec^2(ax+b)dx=\frac{1}{a}\tan(ax+b)+C$.

eg.7. $\int\sin(1-2x)\,dx \xlongequal[du=-2dx]{u=1-2x} \int\sin u\cdot\frac{du}{-2}=\frac{1}{-2}(-\cos u)+C=\boxed{\frac{-1}{-2}\cos(1-2x)+C}$

$\frac{du}{-2}=dx$

③ u-sub for definite integral.

eg.8. Evaluate the definite integral $\int_0^2 \frac{1}{\sqrt{9-4x}}\,dx$.

Ugly part: $9-4x$. $u=9-4x$, $du=-4\cdot dx \Rightarrow dx=\frac{du}{-4}$

Caution: $\int_0^2$ 0, 2 are for x. They also change as we substitute $9-4x$ by u.

$$\int_{x=0}^{x=2} \xrightarrow{u=9-4x} \int_{u=9-4\cdot 0=9}^{u=9-4\cdot 2=1}, \quad \int_{u=9}^{u=1}$$

$$\int_0^2 \frac{1}{\sqrt{9-4x}}\,dx \;\underline{\underline{u=9-4x}}\; \int_9^1 \frac{1}{\sqrt{u}}\cdot\frac{du}{-4}$$ Use the flipping trick

$$= \int_1^9 \frac{1}{4}\cdot u^{-\frac{1}{2}}\,du = \frac{1}{4}\cdot\frac{1}{\frac{1}{2}}u^{\frac{1}{2}}\Big|_1^9$$

$$= \frac{1}{2}\cdot\sqrt{u}\Big|_1^9 = \boxed{\frac{1}{2}\cdot\sqrt{9}-\frac{1}{2}\cdot\sqrt{1}} = \boxed{1}$$

eg.9. Evaluate $\int_0^{\frac{\pi}{2}} 3\cdot\tan(\frac{x}{2})\cdot\sec^2(\frac{x}{2})\,dx$. Hint: $(\tan\blacksquare)' = \sec^2\blacksquare$

$u=\tan(\frac{x}{2})$. $du=\frac{1}{2}\sec^2(\frac{x}{2})\,dx$. $\int_{x=0}^{x=\frac{\pi}{2}} \longrightarrow \int_{u=\tan 0=0}^{u=\tan(\frac{\pi}{4})=1}$

$2du = \sec^2(\frac{x}{2})\,dx$.

$$= \int_0^1 3\cdot u\cdot 2du = 6\cdot\frac{1}{2}u^2\Big|_0^1 = \boxed{3\cdot 1^2-3\cdot 0^2=3}$$

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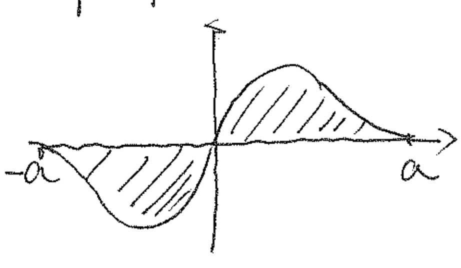
④ Symmetry integral.

If $f(x)$ is odd, i.e., $f(-x)=-f(x)$, then $\int_{-a}^{a} f(x)\,dx=0$



f odd means the graph of f is symmetric about the origin. Then the area above and below x-axis are the same, which will be cancelled out.

eg.10. $f(x)=\sin x\cdot(x^2+1)$. $\int_{-8}^{8}\sin x\cdot(x^2+1)\,dx=0$

since $f(-x)=\sin(-x)\cdot((-x)^2+1)=-\sin x\cdot(x^2+1)=-f(x)$ ($\sin x$ is odd)

More examples and hints for webwork:

eg 11. (ww8). (14 Final) $\int_8^{11} x\cdot\sqrt{x-7}\,dx$. Hint: $u=x-7$. $du=dx$.

$= \int x\cdot\sqrt{u}\,du$ — there is still one x left. keep substituting via the relation $u=x-7 \iff u+7=x$.

$= \int_1^4 (u+7)\cdot\sqrt{u}\cdot du$ — $x=11 \to u=x-7=4$, $x=8 \to u=x-7=1$

$= \int_1^4 u\cdot u^{\frac12}+7\cdot u^{\frac12}\,du$

$= \int_1^4 u^{\frac32}+7\cdot u^{\frac12}\,du = \frac25 u^{\frac52}+7\cdot\frac23 u^{\frac32}\Big|_1^4$

$4^{\frac52}=(\sqrt4)^5=32$.

$4^{\frac32}=(\sqrt4)^3=8$

$= \frac25\cdot 4^{\frac52}+\frac{14}{3}\cdot 4^{\frac32}-\left(\frac25\cdot 1+\frac{14}{3}\cdot 1\right)$

$= \frac{64}{5}+\frac{112}{3}-\frac25-\frac{14}{3} = \boxed{\frac{62}{5}+\frac{98}{5}}$

eg 12. (ww6) $\int \frac{1}{x^2}\sin\left(\frac3x\right)\cdot\cos\left(\frac3x\right)dx$. Hint: $u=\cos\left(\frac3x\right)$.

$du=-\sin\left(\frac3x\right)\cdot\left(\frac3x\right)'\,dx$. chain rule. inner $\frac3x=3x^{-1}$

$=-\sin\left(\frac3x\right)\cdot\frac{-3}{x^2}\,dx$ — $(3\cdot x^{-1})'=3\cdot\frac{-1}{x^2}$

$=\sin\left(\frac3x\right)\cdot\frac{3}{x^2}\,dx$.

$\Rightarrow \frac13 du=\sin\left(\frac3x\right)\cdot\frac{1}{x^2}\,dx$

$= \int \cos\left(\frac3x\right)\cdot\frac{1}{x^2}\sin\left(\frac3x\right)dx$

$= \int u\cdot\frac13 du$

$= \frac13\cdot\frac12 u^2 + C$

$= \boxed{\frac16\cdot\left(\cos\left(\frac3x\right)\right)^2+C}$

eg 13. (14 Final) $\int \frac{x^3}{\sqrt{1+2\cdot x^4}}\,dx$. Ugly part: $u=1+2x^4$, $du=8\cdot x^3\,dx$

$\Rightarrow \frac18 du = x^3\,dx$

$= \int \frac{1}{\sqrt{1+2x^4}}\cdot x^3\,dx$

$= \int \frac{1}{\sqrt u}\cdot\frac18 du = \int u^{-\frac12}\cdot\frac18 du = \frac{1}{-\frac12+1}u^{-\frac12+1}\cdot\frac18+C$

$=2(u)^{\frac12}\cdot\frac18+C$

$=\frac14\cdot(1+2x^4)^{\frac12}+C$.

1 Properties of Integral

1.
$$\int_a^b f(x)\mathrm{d}x = -\int_b^a f(x)\mathrm{d}x, \quad \int_a^a f(x)\mathrm{d}x = 0$$

2.
$$\int_a^b \text{CONSTANT} \;\; \mathrm{d}x = \text{CONSTANT} \cdot (b-a)$$

3.
$$\int_a^b [f(x)+g(x)]\mathrm{d}x = \int_a^b f(x)\mathrm{d}x + \int_a^b g(x)\mathrm{d}x, \quad \int_a^b cf(x)\mathrm{d}x = c\int_a^b f(x)\mathrm{d}x$$

4.
$$\int_a^b [c_1 f(x) + c_2 g(x)]\mathrm{d}x = c_1\int_a^b f(x)\mathrm{d}x + c_2\int_a^b g(x)\mathrm{d}x$$

5.
$$\boxed{\int_a^b f(x)\mathrm{d}x = \int_a^c f(x)\mathrm{d}x + \int_c^b f(x)\mathrm{d}x}$$

6.
   - If $f(x) \geq 0$, then $\int_a^b f(x)\mathrm{d}x \geq 0$.
   - If $f(x) \geq g(x)$, then $\int_a^b f(x)\mathrm{d}x \geq \int_a^b g(x)\mathrm{d}x$.
   - If $m \leq f(x) \leq M$, then $m(b-a) \leq \int_a^b f(x)\mathrm{d}x \leq M(b-a)$

2 Fundamental Theorem of Calculus (FTC)

1. Differential Rule:

$$\text{If} \quad g(x) = \int_a^x f(t)\mathrm{d}t, \quad \text{then} \quad g'(x) = \frac{\mathrm{d}}{\mathrm{d}x}g(x) = f(x)$$

$$\boxed{\text{If} \quad g(x) = \int_{v(x)}^{u(x)} f(t)\mathrm{d}t, \quad \text{then} \quad g'(x) = \frac{\mathrm{d}}{\mathrm{d}x}g(x) = f[u(x)]\cdot u'(x) - f[v(x)]\cdot v'(x)}$$

2.
$$\int_a^b f(x)\mathrm{d}x = F(x)|_a^b = F(b) - F(a), \quad \text{where} \quad F'(x) = f(x)$$

Math133-Table of (Indefinite) Integral

$$\int cf(x)\mathrm{d}x = c\int f(x)\mathrm{d}x \qquad \int [f(x)+g(x)]\mathrm{d}x = \int f(x)\mathrm{d}x + \int g(x)\mathrm{d}x$$

$$\boxed{\int k\mathrm{d}x = kx + C} \qquad \int_a^b k\mathrm{d}x = k(b-a)$$

$$\boxed{\int x^n\mathrm{d}x = \frac{x^{n+1}}{n+1} + C \ (n \neq -1)} \qquad \boxed{\int \frac{1}{x}\mathrm{d}x = \ln|x| + C}$$

$$\int e^x\mathrm{d}x = e^x + C \qquad \boxed{\int b^x\mathrm{d}x = \frac{b^x}{\ln b} + C}$$

$$\boxed{\int \sin x\mathrm{d}x = -\cos x + C} \qquad \boxed{\int \cos x\mathrm{d}x = \sin x + C}$$

$$\boxed{\int \sec^2 x\mathrm{d}x = \tan x + C} \qquad \int \csc^2 x\mathrm{d}x = -\cot x + C$$

$$\int \sec x \tan x\mathrm{d}x = \sec x + C \qquad \int \csc x \cot x\mathrm{d}x = -\csc x + C$$

$$\boxed{\int \frac{1}{1+x^2}\mathrm{d}x = \tan^{-1} x + C} \qquad \int \frac{1}{\sqrt{1-x^2}}\mathrm{d}x = \sin^{-1} x + C$$

$$\int \sinh x\mathrm{d}x = \cosh x + C \qquad \int \cosh x\mathrm{d}x = \sinh x + C$$