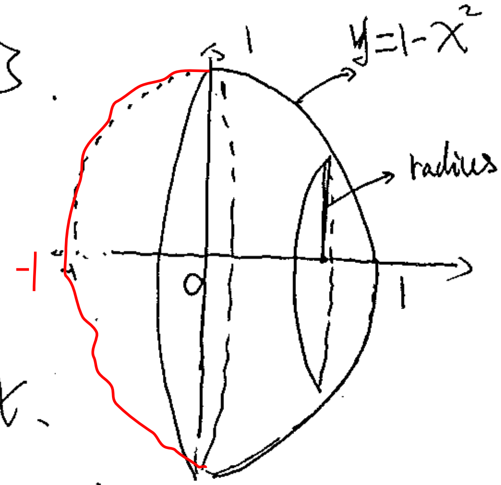
Volume ~~§6.2~~ §5.2 Solutions to Exercises

2, 5, 7, 8, 10, 11, 13, 14, 16

2. $y=1-x^2$, $y=0$; about x-axis.



sln: disk, radius (is height)

$y=1-x^2$

about x axis, then integrate w.r.t. x.

from ~~$x=0$~~ $x=-1$, to $x=1$

$$V=\int_{\cancel{0}\,-1}^{1}\pi\,(\text{radius})^2\,dx=\int_{\cancel{0}\,-1}^{1}\pi\,(1-x^2)^2\,dx=\int_{\cancel{0}\,-1}^{1}\pi\,(1-2x^2+x^4)\,dx$$

$$=\pi\left(x-\tfrac{2}{3}x^3+\tfrac{1}{5}x^5\right)\Big|_{\cancel{0}\,-1}^{1}$$

$$=\cancel{\pi\left(1-\tfrac{2}{3}\cdot 1^3+\tfrac{1}{5}\cdot 1^5\right)-\pi\cdot 0}$$

$$=\pi\cdot\frac{8}{15}+\pi\cdot\frac{8}{15}$$

$$=\pi\,\frac{16}{15}$$

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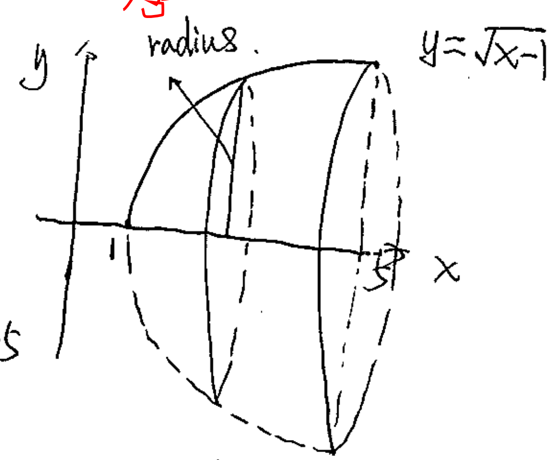
3. $y=\sqrt{x-1}$, $y=0$, $x=5$; about x axis.



sln: disk, radius (is height)

$y=\sqrt{x-1}$, integrate w.r.t. x.

from $x=1$ to $x=5$

$$V=\int_1^5\pi\left(\sqrt{x-1}\right)^2dx$$

$$=\int_1^5\pi\,(x-1)\,dx$$

$$=\pi\left(\tfrac{1}{2}x^2-x\right)\Big|_1^5$$

$$=\pi\left(\tfrac{1}{2}\cdot 5^2-5\right)-\pi\cdot\left(\tfrac{1}{2}\cdot 1^2-1\right)$$

$$=8\pi$$

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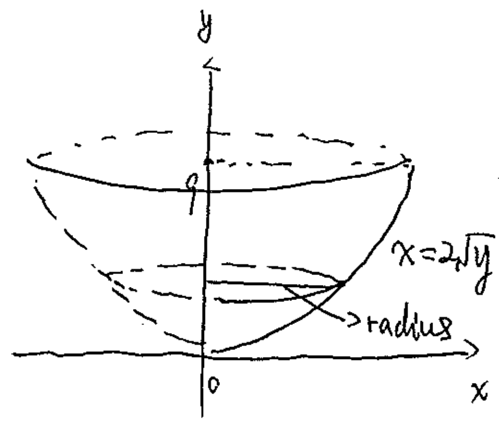
5. $x = 2\sqrt{y}$, $x=0$, $y=9$; about y-axis



sln: $(x^2 = 4y)$, disk.

about y-axis, integrate w.r.t. y

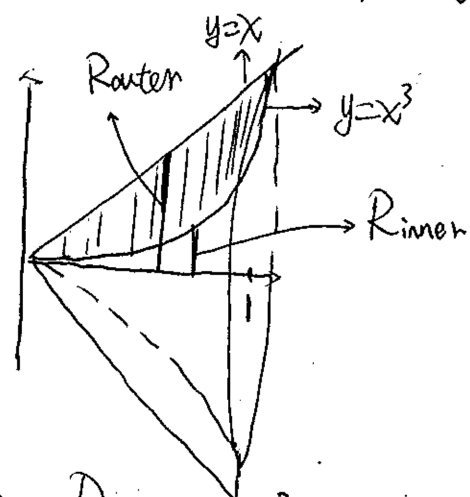
radius: $2\sqrt{y}$ from $y=0$ to $y=9$

$$V = \int_0^9 \pi\,(\text{radius})^2\,dy = \int_0^9 \pi\cdot(2\sqrt{y})^2\,dy = \int_0^9 4\pi\, y\,dy$$

$$= 4\pi\cdot\frac{1}{2}y^2\Big|_0^9 = 4\pi\cdot\frac{1}{2}\cdot 9^2 - 0$$

$$= 162\pi$$

7. $y = x^3$, $y = x$, $x \geq 0$; about x-axis.



sln: washer;

about x axis, integrate w.r.t. x.

intersection points

$$\begin{cases} y = x^3 \\ y = x \end{cases} \Rightarrow x^3 = x \Rightarrow x = 0 \text{ or } x = 1$$

outer radius: $R_{outer} = x - 0$ inner radius: $R_{inner} = x^3 - 0$

$$V = \int_0^1 \pi\left[R_{outer}^2 - R_{inner}^2\right]dx$$

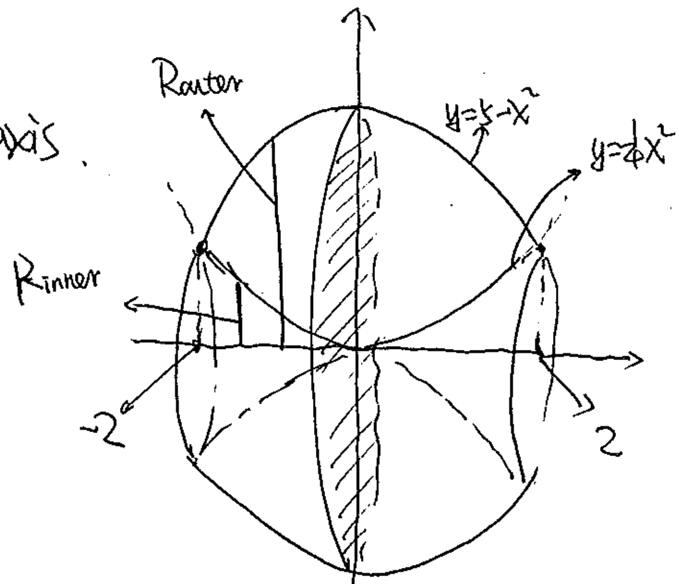
$$= \int_0^1 \pi\left[x^2 - (x^3)^2\right]dx$$

$$= \int_0^1 \pi(x^2 - x^6)\,dx$$

$$= \pi\left(\frac{1}{3}\cdot x^3 - \frac{1}{7}x^7\right)\Big|_0^1$$

$$= \pi\left(\frac{1}{3}\cdot 1^3 - \frac{1}{7}\cdot 1^7\right) - \pi\cdot 0 = \pi\,\frac{4}{21}$$

8. $y=\frac{1}{4}x^2$, $y=5-x^2$; about x-axis.



sln: ~~integrate~~ washer:

about x-, then integrate w.r.t. x.

intersection points

$$\begin{cases} y=\frac{1}{4}x^2 \\ y=5-x^2 \end{cases} \Rightarrow \frac{1}{4}x^2=5-x^2 \Rightarrow \frac{1}{4}x^2+x^2=5$$

$$\Rightarrow \frac{5}{4}x^2=5 \Rightarrow x^2=4 \Rightarrow x=-2 \text{ or } x=2$$

integrate x from $x=-2$ to $x=2$

$R_{outer} = 5-x^2-0 = 5-x^2$, $\quad R_{inner}=\frac{1}{4}x^2$

$$V=\int_{-2}^{2}\pi\left[R_{outer}^2-R_{inner}^2\right]dx=\int_{-2}^{2}\pi\left[(5-x^2)^2-\left(\frac{1}{4}x^2\right)^2\right]dx$$

$$=\int_{-2}^{2}\pi\left(25-10x^2+x^4-\frac{1}{16}x^4\right)dx$$

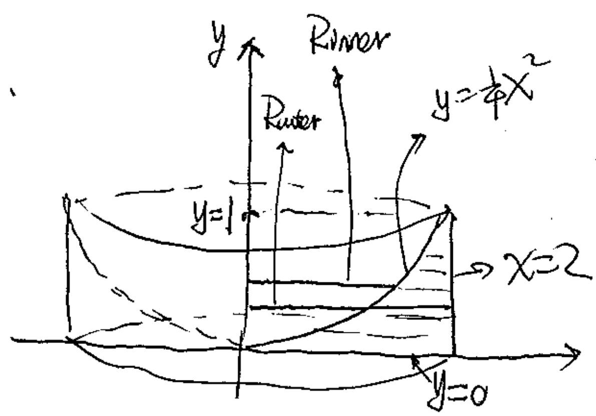
$$=\int_{-2}^{2}\pi\cdot\left(25-10x^2+\frac{15}{16}x^4\right)dx$$

$$=\pi\left(25x-\frac{10}{3}x^3+\frac{15}{16}\cdot\frac{1}{5}x^5\right)\Big|_{-2}^{2}$$

$$=\pi\left(25\times2-\frac{10}{3}\times2^3+\frac{3}{16}\cdot2^5\right)-\pi\left[25\cdot(-2)-\frac{10}{3}(-2)^3+\frac{3}{16}(-2)^5\right]$$

$$=176\pi$$

10 $y=\frac{1}{4}x^2$, $x=2$, $y=0$; about y.



sln: washer

about y. then integrate w.r.t. y

intersection points (solve y)

$$\begin{cases} y=\frac{1}{4}x^2 \Rightarrow 4y=x^2 \Rightarrow \sqrt{4y}=x \Rightarrow \begin{cases} x=\sqrt{4y} \\ x=2 \end{cases} \\ x=2. \end{cases}$$

$\sqrt{4y}=2 \Rightarrow 4y=4 \Rightarrow y=1, \Rightarrow$ integrate w.r.t. y from $y=0$ to $y=1$.

$R_{outer}=2$, $R_{inner}=\sqrt{4y}$

$$V=\int_0^1 \pi\left(2^2-(\sqrt{4y})^2\right)dy=\int_0^1 \pi(4-4y)dy$$
$$=\pi\left(4y-4\cdot\frac{1}{2}y^2\right)\Big|_0^1$$
$$=\pi(4\cdot 1-2\cdot 1^2)-\pi\cdot 0=2\pi$$

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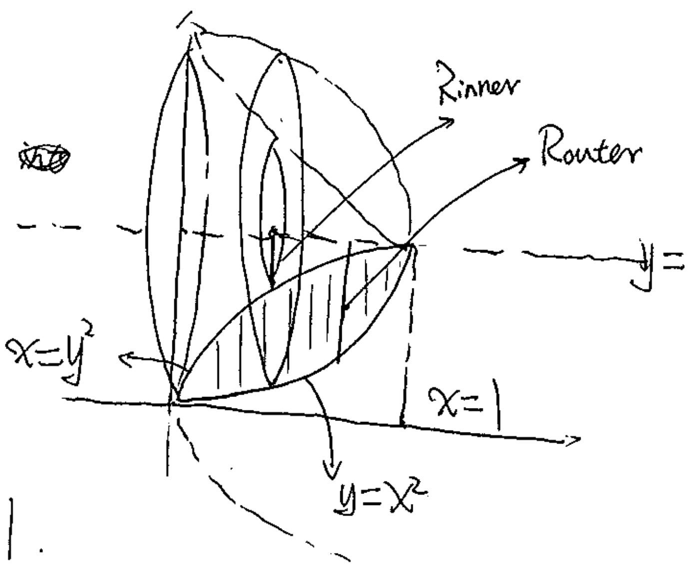
11. $y=x^2$, $x=y^2$; about $y=1$



s/n: about $y=1$ ⇒ (horizontal) ⇒

⇒ integrate w.r.t. x.

intersection points:

$$\begin{cases} y=x^2 \\ x=y^2 \Rightarrow y=\sqrt{x} \end{cases} \Rightarrow x^2=\sqrt{x} \Rightarrow x=0, \text{ or } 1.$$

integrate w.r.t. x from $x=0$ to $x=1$

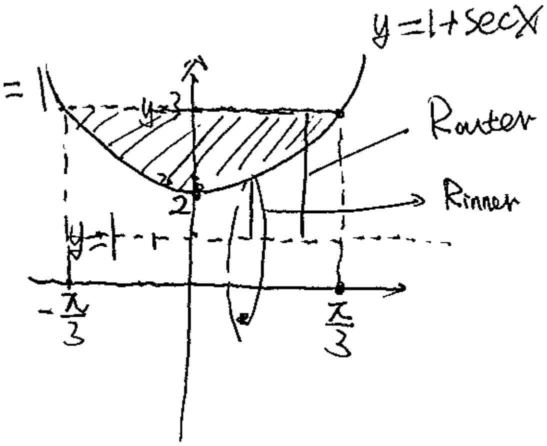
$R_{outer}=x^2-1$, $R_{inner}=\sqrt{x}-1$

$$V=\int_0^1 \pi\left[(x^2-1)^2-(\sqrt{x}-1)^2\right]dx$$
$$=\int_0^1 \pi\left[x^4-2x^2+1-\left((\sqrt{x})^2-2\sqrt{x}+1\right)\right]dx$$
$$=\int_0^1 \pi(x^4-2x^2-x+2\sqrt{x})dx$$

$$=\pi\left(\frac{1}{5}x^5-\frac{2}{3}x^3-\frac{1}{2}x^2+2\frac{1}{2}x^{-\frac{1}{2}}\right)\Big|_0^1$$
$$=\pi\left(\frac{1}{5}-\frac{2}{3}-\frac{1}{2}+1\right)$$
$$=\pi\cdot\frac{11}{30}$$

※

13 $y = 1 + \sec x$, $y = 3$; about $y = 1$



s/n: about $y=1$ (horizontal)
then integrate w.r.t. x.
solve intersection points:

$$\begin{cases} y = 1+\sec x \\ y = 3 \end{cases} \Rightarrow 1+\sec x = 3 \Rightarrow \sec x = 2 \Rightarrow \frac{1}{\cos x} = 2$$

$$\Rightarrow \cos x = \frac{1}{2} \Rightarrow x = \frac{\pi}{3} \text{ or } x = -\frac{\pi}{3}$$

integrate w.r.t. x from $x = \cancel{0}\ -\frac{\pi}{3}$ to $x = \frac{\pi}{3}$

$R_{outer} = 3 - 1 = 2$. $R_{inner} = (1+\sec x) - 1 = \sec x$

$$V = \int_{-\frac{\pi}{3}}^{\frac{\pi}{3}} \pi [\, 2^2 - \sec^2 x] \, dx = \pi (4x - \tan x) \Big|_{-\frac{\pi}{3}}^{\frac{\pi}{3}}$$

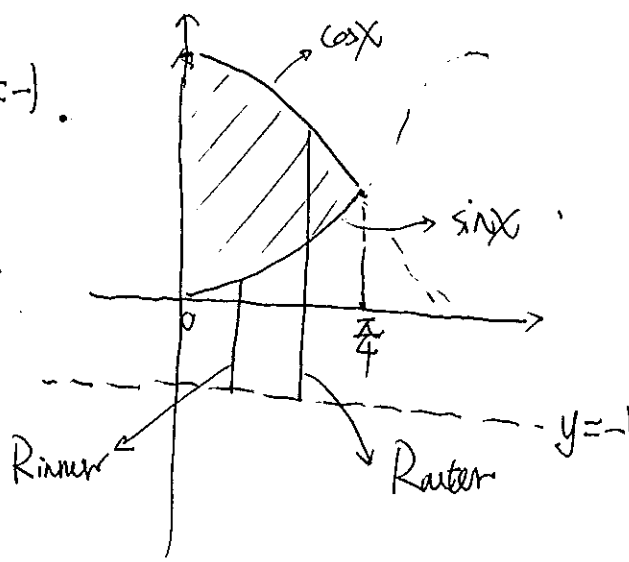
Hint: $\tan\frac{\pi}{3} = \sqrt{3}$, $\tan(-\frac{\pi}{3}) = -\sqrt{3}$

$$= \pi(4 \times \frac{\pi}{3} - \sqrt{3}) - \pi(4(-\frac{\pi}{3}) - (-\sqrt{3}))$$

$$= \pi(\frac{8\pi}{3} - 2\sqrt{3})$$

$$= 2\pi(\frac{4}{3}\pi - \sqrt{3})$$

14. $y = \sin x$, $y = \cos x$, $0 \le x \le \frac{\pi}{4}$; about $y = -1$.



s/n: about $y = -1$ (horizontal)
integrate w.r.t. x from $x=0$ to $x = \frac{\pi}{4}$

$R_{outer} = \cos x - (-1)$; $R_{inner} = \sin x - (-1)$
$= \cos x + 1$ $\qquad = \sin x + 1$

$$V = \int_0^{\frac{\pi}{4}} \pi [(\cos x + 1)^2 - (\sin x + 1)^2] \, dx$$

$$= \int_0^{\frac{\pi}{4}} \pi [\cos^2 x + 2\cdot\cos x + 1 - (\sin^2 x + 2\sin x + 1)] \, dx$$

$$= \int_0^{\frac{\pi}{4}} \pi \left[\cos^2 x + 2\cos x - \sin^2 x - 2\sin x \right] dx$$

$$= \int_0^{\frac{\pi}{4}} \pi (\cos^2 x - \sin^2 x) dx + \int_0^{\frac{\pi}{4}} \pi (2\cos x - 2\sin x) dx.$$

$$\int_0^{\frac{\pi}{4}} \pi (\cos^2 x - \sin^2 x) dx = \int_0^{\frac{\pi}{4}} \pi \cdot \cos 2x \, dx$$

Hint: $\cos^2 x - \sin^2 x = \cos 2x$.

$$\underset{du = 2dx}{\overset{u = 2x}{=\!=}} \int_{u(0)}^{u(\frac{\pi}{4})} \pi \cdot \cos u \cdot \frac{du}{2}$$

$$= \int_0^{\frac{\pi}{2}} \frac{\pi}{2} \cos u \cdot du$$

$$= \frac{\pi}{2} \cdot \sin u \Big|_0^{\frac{\pi}{2}} = \frac{\pi}{2} \cdot \sin\frac{\pi}{2} - \frac{\pi}{2} \sin 0$$

$$= \frac{\pi}{2}.$$

$$= \frac{\pi}{2} + 2\pi \int_0^{\frac{\pi}{4}} (\cos x - \sin x) dx$$

$$= \frac{\pi}{2} + 2\pi \cdot (\sin x + \cos x) \Big|_0^{\frac{\pi}{4}}$$

$$= \frac{\pi}{2} + 2\pi \left(\sin\frac{\pi}{4} + \cos\frac{\pi}{4}\right) - 2\pi (\sin 0 + \cos 0)$$

$$= \frac{\pi}{2} + 2\pi \left(\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}\right) - 2\pi \cdot 1$$

$$= \frac{\pi}{2} + 2\pi \cdot \sqrt{2} - 2\pi$$

$$= -\frac{3}{2}\pi + 2\pi\sqrt{2}$$

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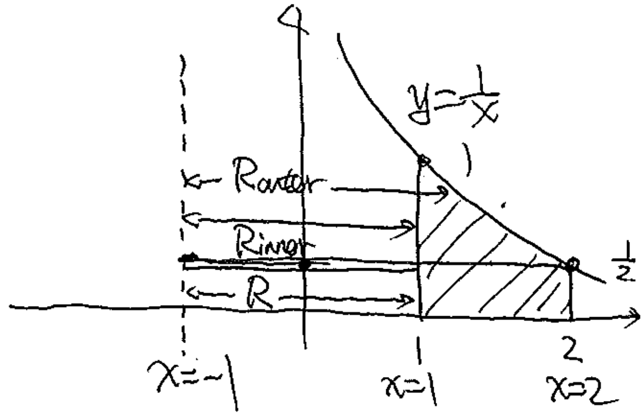
16. $xy=1$, $y=0$, $x=1$, $x=2$; about $x=-1$

sln: $xy=1 \Leftrightarrow x=\frac{1}{y}$



$x=2$ intersects $xy=1$ at $y=\frac{1}{2}$

from $y=0$ to $y=\frac{1}{2}$

$$V_1 = \int_0^{\frac{1}{2}} \pi\left[(2-(-1))^2-(1-(-1))^2\right]dx$$

$$= \int_0^{\frac{1}{2}} \pi\left[3^2-2^2\right]dx = \int_0^{\frac{1}{2}} \pi\cdot 5\,dx = 5\pi\cdot\left(\frac{1}{2}-0\right)$$

$$= \frac{5}{2}\pi.$$

$x=1$ intersect xy at $y=1$

from $y=\frac{1}{2}$ to $y=1$

$R_{outer} = \frac{1}{y}-(-1) = \frac{1}{y}+1$, $R_{inner} = 1-(-1) = 2$

$$V_2 = \int_{\frac{1}{2}}^{1} \pi\left[\left(\frac{1}{y}+1\right)^2 - 2^2\right]dy$$

$$= \int_{\frac{1}{2}}^{1} \pi\left(\frac{1}{y^2}+\frac{2}{y}+1-4\right)dy$$

$$= \int_{\frac{1}{2}}^{1} \pi\left(y^{-2}+2\cdot\frac{1}{y}-3\right)dy$$

$$= \pi\cdot\left(-y^{-1}+2\ln y-3y\right)\Big|_{\frac{1}{2}}^{1} = \pi(-1+2\ln 1-3)-\pi\left(-\left(\tfrac{1}{2}\right)^{-1}+2\ln\tfrac{1}{2}-3\tfrac{1}{2}\right)$$

$$= \pi(-4)-\pi\left(-\frac{7}{2}-2\ln 2\right)$$

$$= \pi\left(2\ln 2-\frac{1}{2}\right)$$

$$V = V_1+V_2 = \pi(2\ln 2+2)$$