Landscape theory for finite $\mathbb{Z}^1$ lattice,

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**Theorem 0.1** (Landscape theory for finite matrix. M. L. Lyra, S. Mayboroda and M. Filoche).

Let

$$H = \begin{pmatrix} v_1 & -1 & 0 & \cdots & 0 \\ -1 & v_2 & -1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & -1 & v_n \end{pmatrix}. \quad (0.1)$$

If $v_j \geq 2, j = 1, \ldots, n$, and $H\vec{x} = \lambda\vec{x}$, then for all $j = 1, \ldots, n$,

$$\frac{|x_j|}{\max_{1 \leq k \leq n}|x_k|} \leq \lambda u_j, \quad (0.2)$$

where $\vec{u} \in \mathbb{R}^n$ is the landscape function satisfying

$$H\vec{u} = \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix} =: \vec{1}. \quad (0.3)$$

1 Useful lemmas

**Lemma 1.1.** Let $H$ be given in (0.1). If $v_j \geq 2, j = 1, \ldots, n$, then $H$ is invertible. Let $G_{ij} = H^{-1}(i,j)$ be the $(i,j)$ entry of the inverse of $H$. As a consequence, there is always a $\vec{u} \in \mathbb{R}^n$ satisfying eq. (0.3), with explicit expression as

$$u_j = \sum_{k=1}^n G_{jk}. \quad (1.1)$$

Moreover, all the eigenvalues of $H$ are strictly positive.

**Lemma 1.2.** Let $H$, $G_{ij} = H^{-1}(i,j)$ and $u_j$ be as above. $G_{ij} > 0$ for all $i, j$. As a consequence,

$$u_j > 0. \quad (1.2)$$

**Exercise 1.3.** Use Lemma 1.2 to prove Theorem 0.2.
2 Proof of Lemma 1.1: Existence of Green’s function

Let \( H = V - H_0 \), where

\[
V = \begin{pmatrix}
v_1 & 0 & 0 & \cdots & 0 \\
0 & v_2 & 0 & \ddots & : \\
0 & \ddots & \ddots & \ddots & 0 \\
\vdots & \ddots & \ddots & \ddots & \vdots \\
0 & \cdots & 0 & v_{n-1} & 0 \\
0 & \cdots & 0 & 0 & v_n
\end{pmatrix}, \quad \text{and} \quad
H_0 = \begin{pmatrix}
0 & 1 & 0 & \cdots & 0 \\
1 & 0 & 1 & \ddots & : \\
0 & \ddots & \ddots & \ddots & 0 \\
\vdots & \ddots & \ddots & \ddots & \vdots \\
0 & \cdots & 0 & 1 & 0
\end{pmatrix}
\tag{2.1}
\]

**Exercise 2.1.** Consider the usual matrix 2-norm \( \| \cdot \| \) (defined in HW1). Prove that \( \| H_0 \| \leq 2 \). As a consequence, all eigenvalues \( \{ \mu_1, \mu_2, \cdots, \mu_n \} \) of \( H_0 \) are contained in \([-2, 2]\), i.e. \( |\mu_j| \leq 2 \) for all \( j \).

**Proof.** Let

\[
R = \begin{pmatrix}
0 & 0 & 0 & \cdots & 0 \\
1 & 0 & 0 & \ddots & : \\
0 & \ddots & \ddots & \ddots & 0 \\
\vdots & \ddots & \ddots & \ddots & \vdots \\
0 & \cdots & 0 & 1 & 0
\end{pmatrix}, \quad
L = \begin{pmatrix}
0 & 1 & 0 & \cdots & 0 \\
0 & 0 & 1 & \ddots & : \\
0 & \ddots & \ddots & \ddots & 0 \\
\vdots & \ddots & \ddots & \ddots & \vdots \\
0 & \cdots & 0 & 0 & 1
\end{pmatrix}
\tag{2.2}
\]

Clearly, \( H_0 = L + R \). For any \( \vec{x} = (x_1, x_2, \cdots, x_n)^T \), direct computation shows that

\[
R \vec{x} = R = \begin{pmatrix}
x_1 \\
x_2 \\
\vdots \\
x_{n-1} \\
x_n
\end{pmatrix} = \begin{pmatrix}
0 \\
x_1 \\
\vdots \\
x_{n-2} \\
x_{n-1}
\end{pmatrix}
\tag{2.3}
\]

Therefore,

\[
\| R \vec{x} \|^2 = x_1^2 + x_2^2 + \cdots + x_{n-1}^2 \leq x_1^2 + x_2^2 + \cdots + x_{n-1}^2 + x_n^2 = \| \vec{x} \|^2
\tag{2.4}
\]

which implies \( \| R \vec{x} \| \leq \| \vec{x} \| \). According to the definition of the matrix norm and (2.4):

\[
\| R \| = \max_{\vec{x} \neq 0} \frac{\| R \vec{x} \|}{\| \vec{x} \|} \leq 1.
\tag{2.5}
\]

Exact the same argument shows that \( \| L \| \leq 1 \). Therefore, by the property (triangle inequality) of the matrix norm, we have that

\[
\| H_0 \| \leq \| L \| + \| R \| \leq 2,
\tag{2.6}
\]

which completes the proof.

**Exercise 2.2.** Assume first that \( v_j > 2, j = 1, \cdots, n \), (all \( v_j \) are strictly greater than 2). Prove that \( H = V - H_0 \) is invertible.

**Exercise 2.3.** Prove that \( |\mu_j| < 2 \), i.e., \(-2\) and \( 2 \) are not eigenvalues of \( H_0 \).

*Hint: consider the difference equation \( H_0 \vec{x} = 2 \vec{x} \), where \( \vec{x} = (x_1, \cdots, x_n) \), as part of the infinite system (see Ex. (2.6) below), with zero boundary condition \( x_0 = x_{n+1} = 0 \).*
Exercise 2.4. Prove that \(\|H_0\| < 2\).

*Hint:* prove that for any symmetric \(n \times n\) matrix \(A\)

\[
\|A\| := \sup_{\|\vec{x}\| = 1} \|A\vec{x}\| = \max_{1 \leq j \leq n} |\mu_j|,
\]

where \(\{\mu_1, \mu_2, \cdots, \mu_n\}\) are the eigenvalues of \(A\).

Exercise 2.5. Complete the proof of Lemma 1.1 under the assumption \(v_j \geq 2, j = 1, \cdots, n\).

Exercise 2.6. (Supplementary problem) Consider the difference equation (on an infinite lattice)

\[
x_{n+1} + x_{n-1} = \lambda x_n, \quad n \in \mathbb{Z}, \quad \lambda \in \mathbb{C} \tag{2.8}
\]

1. Prove that the following expression solves eq. (2.8) for all \(n \in \mathbb{Z}\)

\[
x_n = c_1 \mu^n + c_2 \mu^{-n}, \quad c_1, c_2 \in \mathbb{C}, \quad \mu = \frac{\lambda + \sqrt{\lambda^2 - 4}}{2} \tag{2.9}
\]

And find \(c_1, c_2\) (in terms of \(\mu\)) if \(x_0 = 0, x_1 = 1\).

2. For any \(|\lambda| \neq 2\), consider the two infinite sequences \(\vec{\alpha}, \vec{\beta}\) given by \(\mu^n\) and \(\mu^{-n}\), i.e.,

\[
\vec{\alpha} = (\cdots, \alpha_{-1}, \alpha_0, \alpha_1, \alpha_2, \cdots) = (\cdots, \mu^{-1}, 1, \mu^1, \mu^2, \cdots) \tag{2.10}
\]

\[
\vec{\beta} = (\cdots, \beta_{-1}, \beta_0, \beta_1, \beta_2, \cdots) = (\cdots, \mu^1, 1, \mu^{-1}, \mu^{-2}, \cdots) \tag{2.11}
\]

Prove that

\[
W(\vec{\alpha}, \vec{\beta}) := \det \begin{pmatrix} \alpha_{n+1} & \beta_{n+1} \\ \alpha_n & \beta_n \end{pmatrix} \tag{2.12}
\]

is a constant (independent of \(n\)). \(^1\)

3. Putting 1 and 2 together, we actually can say that if \(|\lambda| \neq 2\), then all solutions \(\vec{x} = \{x_j\}\) to eq. (2.8) is a linear combination of \(\vec{\alpha}, \vec{\beta}\),

\[
\vec{x} = c_0 \vec{\alpha} + c_1 \vec{\beta} \tag{2.13}
\]

This is not the case if \(|\lambda| = 2\). Prove that if \(|\lambda| = 2\), then \(\vec{\alpha} = \vec{\beta} = a\) constant vector. Then find another solution (a non-constant vector) \(\vec{\gamma} = \{\gamma_n\}\), which solves eq. (2.8) (for \(\lambda = 2\)) and satisfies that \(W(\vec{\alpha}, \vec{\gamma})\) is a constant.

3 Proof of Lemma 1.2: Positivity of the Green’s function

3.1 Maximum principle and the first (original) proof

**Lemma 3.1** (Maximum principle). Let \(H\) be given as in (0.1). For any \(\vec{x} \in \mathbb{R}^n\), let \(\vec{y} = H\vec{x}\). If \(y_i \geq 0\) for all \(i = 1, 2, \cdots, n\), then

\[
x_i \geq 0, \quad i = 1, 2, \cdots, n. \tag{3.1}
\]

\(^1W\) is usually referred to be the Wronski of the system (or simply of eq. (2.8)), which plays important role in the general study of second order difference/differential equations.
Exercise 3.2. Prove Lemma 3.1 by contradiction.

*Hint:* consider the equation \( H\vec{x} = \vec{y} \) as a boundary problem on the extended lattice: \([0, 1, \cdots, n, n+1]\), with zero boundary condition: \( x_0 = x_{n+1} = 0 \). Fix \( \vec{y} \), assume that there is a minimum inside the lattice, that is, there exists \( j \in [1, \cdots, n] \) such that \( x_j \leq x_{j+1} \) and \( x_j \leq x_{j-1} \). Prove that this will contradict the condition that \( y_i \geq 0 \) for all \( i = 1, 2, \cdots, n \).

Lemma 3.3 (Strong Maximum principle). Following the notation in Lemma 3.1 if we assume additionally that there exist an \( i_0 \) such that \( y_{i_0} > 0 \) (strictly positive), then

\[
x_i > 0, \quad i = 1, 2, \cdots, n.
\]

(3.2)

Exercise 3.4. Prove Lemma 3.3 by contradiction.

*Hint:* continue with Lemma 3.1, assume that there exists \( j \in [1, \cdots, n] \) such that \( x_j = 0 \). Prove that this will lead to \( y_i \geq 0 \) for all \( i = 1, 2, \cdots, n \), which contradict the condition that \( y_{i_0} > 0 \) for some \( i_0 \).

Lemma 3.5. Let \( \vec{e}_j = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \), \( j = 1, 2, \cdots, n \) be the standard basis of \( \mathbb{R}^n \), with 0 entries except for 1 in the \( j \)-th place. Let

\[
\vec{g} = \begin{pmatrix} G_{1j} \\ G_{2j} \\ \vdots \\ G_{nj} \end{pmatrix}
\]

be the \( j \)-th column vector of \( H^{-1} \), where \( H^{-1} = \{ G_{ij} \} \) is given as in Lemma 1.1. Prove that for all \( j = 1, 2, \cdots, n \),

\[
H\vec{g} = \vec{e}_j
\]

(3.4)

Exercise 3.6. Use Lemma 3.3 and Lemma 3.5 to prove Lemma 1.2.

*Hints:* apply Lemma 3.5 to each pair of \( \vec{g} \) and \( \vec{e}_j \).

3.2 Power series expansion and an alternative proof of Lemma 1.2

References
