Landscape Theory for Tight-Binding Hamiltonians

Undergraduate Math/Stat Research Project Introductions

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Introduction and Project Outline

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- Anderson localization: one of the central phenomena studied in modern mathematical physics, especially in dimensions 2 and 3, starting from Nobel-prize winning discovery by P. W. Anderson.
- Landscape theory: a new theory unveils a direct relationship between the random potential and the localized states.
- Goals of the project:
  - Generalization of landscape theory in the discrete lattice (based only on linear algebra knowledge)
  - Numerical experiments: complexity of computational problems
  - Landscape theory on higher dimensional space and infinite lattices

Schrödinger Matrix and its Eigenvalue Problem

- An example of 4×4 Schrödinger matrix:

$$H = \begin{pmatrix} v_1 & -1 & 0 & 0 \\ -1 & v_2 & -1 & 0 \\ 0 & -1 & v_3 & -1 \\ 0 & 0 & -1 & v_4 \end{pmatrix}, \quad v_1, v_2, v_3, v_4 \in \mathbb{R}.$$

- Eigenvalue problem of H, $H\vec{x} = \lambda\vec{x}, \lambda \in \mathbb{R}, \vec{x} \in \mathbb{R}^4$:

$$\begin{pmatrix} v_1 & -1 & 0 & 0 \\ -1 & v_2 & -1 & 0 \\ 0 & -1 & v_3 & -1 \\ 0 & 0 & -1 & v_4 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \lambda \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix}$$

Remark

The λ such that $H\vec{x} = \lambda\vec{x}$ has a non-trivial solution $\vec{x} \in \mathbb{R}^4$ is called an eigenvalue of H, and the non-trivial solution $\vec{x}$ is called the associated eigenvector.

Landscape Theory for Schrödinger Matrices

Yikang Li, MSU

- Landscape function, $\vec{u} \in \mathbb{R}^4$ satisfying:

$$H\vec{u} = \begin{pmatrix} v_1 & -1 & 0 & 0 \\ -1 & v_2 & -1 & 0 \\ 0 & -1 & v_3 & -1 \\ 0 & 0 & -1 & v_4 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$$

- Landscape theory (in $\mathbb{R}^4$): If $v_j \geq 2, j = 1, 2, 3, 4$, and $H\vec{x} = \lambda\vec{x}$, then we have

$$\frac{|x_j|}{\max\limits_{k=1,2,3,4} |x_k|} \leq \lambda u_j, \quad j = 1, 2, 3, 4.$$

- The theory holds true for general Schrödinger matrices.

Landscape Theory for $n \times n$ Schrödinger Matrices

Theorem (M. L. Lyra, S. Mayboroda and M. Filoche)

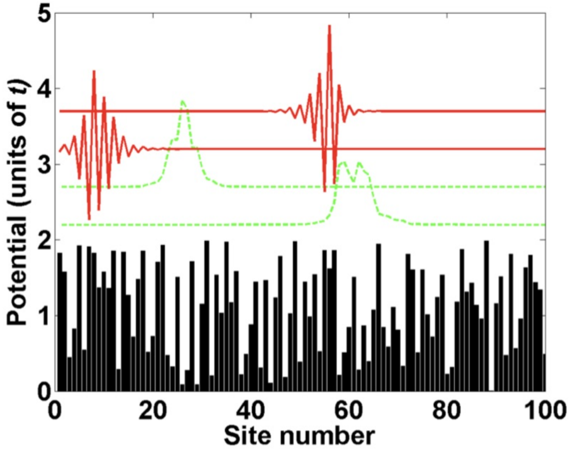
If $H = \begin{pmatrix} v_1 & -1 & 0 & \cdots & 0 \\ -1 & v_2 & -1 & \ddots & \vdots \\ 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & v_{n-1} & -1 \\ 0 & \cdots & 0 & -1 & v_n \end{pmatrix}$, $v_j \geq 2, j = 1, \cdots, n$, *and* $H\vec{x} = \lambda\vec{x}$, *then for all* $j = 1, \cdots, n$,

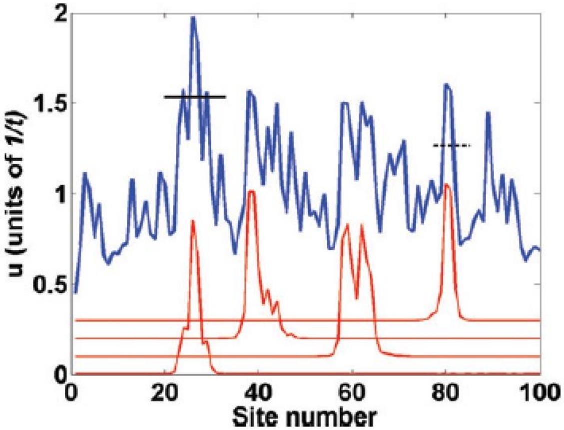
$$\frac{|x_j|}{\max\limits_{1 \leq k \leq n} |x_k|} \leq \lambda u_j, \qquad \textit{where} \quad H\vec{u} = \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix}.$$

- The proof only requires linear algebra knowledge.
- Goal: extending this theory to more general matricies.

Physics and pictures

Isaac Cinzori, MSU



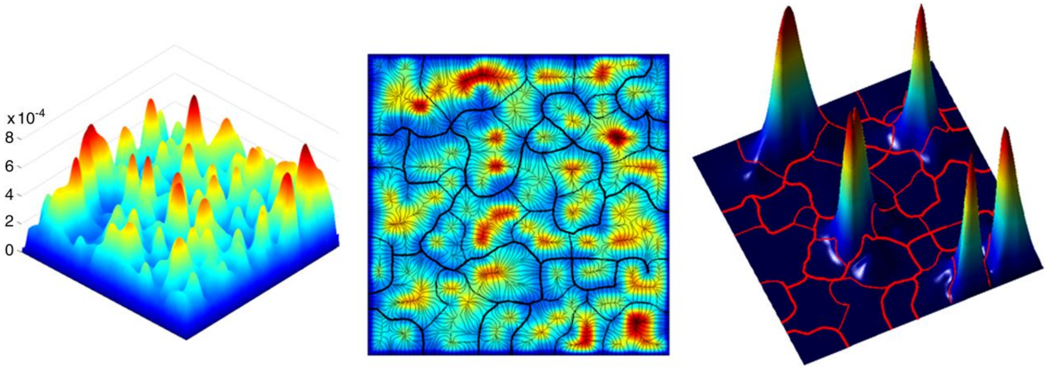


We consider a 100×100 Schrödinger matrix with the potential described by the black graph on the left.

Left figure, green: eigenvectors at small energies. Red: eigenvectors at large energies. Graphs have been shifted vertically.

Right figure, red: low energy eigenvectors. Blue: landscape function.

Physics and pictures: more complicated 2D model



Left and middle figures: plot of the landscape function in a 2D continuous model.

Right figure: Red: contour plot of the "valleys" of the landscape function, together with five low energy eigenvectors.

In both cases, one can conjecture that peaks of landscape function correspond to localization centers of the eigenvectors.

Illustrations are taken from works of M. L. Lyra, S. Mayboroda and M. Filoche.

Thank you!