2019-01-16.

Problem 1. Let A be a $n\times n$ self-adjoint matrix. Prove that

(i) a_{ii} is real for $1\le i\le n$

(ii) let $\lambda_1, \lambda_2, \cdots, \lambda_n$ be the eigenvalues of A. λ_i is real for $1\le i\le n$.

(iii) Suppose $\lambda_i \neq \lambda_j$. let $\vec{\varphi}_i$ and $\vec{\varphi}_j$ be the eigenvectors of λ_i and λ_j respectively. Then $\vec{\varphi}_i \perp \vec{\varphi}_j$.

* (iv) Prove that there is an orthonormal basis $\{\vec{\varphi}_1, \vec{\varphi}_2, \cdots, \vec{\varphi}_n\}$ such that A is diagonalized under this basis if all λ_i are different, i.e.,

$$P^{-1}AP = \begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{pmatrix}, \text{ where } P = (\vec{\varphi}_1, \vec{\varphi}_2, \cdots, \vec{\varphi}_n) \text{ is a } n\times n \text{ matrix.}$$

Remark. (iv) is still true if some eigenvalues are the same. The proof will be slightly more complicated.

Problem 2. For any vector $\vec{x} = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$ in $\mathbb{C}^n$, define the norm of $\vec{x}$ to be

$$\|\vec{x}\| := \sqrt{|x_1|^2 + \cdots + |x_n|^2}$$

Then for a $n\times n$ matrix A, we can define the norm of A by

$$\|A\| := \sup_{\vec{x}\neq 0} \frac{\|A\vec{x}\|}{\|\vec{x}\|}$$

Prove that:

(i) For any $\vec{x}$, $\|A\vec{x}\| \le \|A\|\cdot\|\vec{x}\|$

(ii) $\|A\| = \sup_{\|\vec{y}\|=1} \|A\vec{y}\|$

(iii) If B is also a $n\times n$ matrix, then $\|A+B\| \le \|A\| + \|B\|$.

(iv) Suppose $\|A\| \le 2$, then for all the eigenvalues λ of A, we have $|\lambda| \le 2$.