Sec. 6.3. Diagonalization.

Def. An $n\times n$ matrix A is said to be diagonalizable if there exists a nonsingular matrix X and a diagonal matrix D such that

$$X^{-1}AX = D$$

We say that X diagonalizes A.

Theorem. An $n\times n$ matrix A is diagonalizable if and only if A has n linearly independent eigenvectors.

Pf. Suppose A has n linearly independent eigenvectors $\bar{x}_1, \bar{x}_2, \cdots, \bar{x}_n$. Let $\lambda_1, \lambda_2, \cdots, \lambda_n$ be the eigenvalues corresponding to $\bar{x}_1, \cdots, \bar{x}_n$.

Therefore, $A\bar{x}_i = \lambda_i \cdot \bar{x}_i$ for $i=1,2,\cdots,n$.

Let $X=(\bar{x}_1, \bar{x}_2, \cdots, \bar{x}_n)$ be the $n\times n$ matrix. X is nonsingular since $\bar{x}_1, \cdots, \bar{x}_n$ are linearly independent.

Direct computation shows:

$$A\cdot X = A(\bar{x}_1, \bar{x}_2, \cdots, \bar{x}_n) = (A\bar{x}_1, A\bar{x}_2, \cdots, A\bar{x}_n) = (\lambda_1\bar{x}_1, \cdots, \lambda_n\bar{x}_n)$$

Let $D = \begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{bmatrix}$

$$= (\bar{x}_1, \cdots, \bar{x}_n)\begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{bmatrix}$$

Then $AX = X\cdot D$, which implies

$$\boxed{X^{-1}AX = D}$$

Thm If $\lambda_1, \lambda_2, \cdots, \lambda_k$ are distinct eigenvalues of an $n \times n$ matrix A with corresponding eigenvectors $\bar{x}_1, \cdots, \bar{x}_k$ then $\bar{x}_1, \bar{x}_2, \cdots, \bar{x}_k$ are linearly independent

As a consequence, if A has n distinct eigenvalues, then A is diagonalizable.

- Important formulas:

If $X^{-1} A X = D = \begin{bmatrix} \lambda_1 & & 0 \\ & \lambda_2 & \\ 0 & & \ddots \\ & & & \lambda_n \end{bmatrix}$,

then $A = X \cdot D \cdot X^{-1}$

and $A^k = X \cdot D^k \cdot X^{-1} = X \cdot \begin{bmatrix} (\lambda_1)^k & & 0 \\ & (\lambda_2)^k & \\ 0 & & \ddots \\ & & & (\lambda_n)^k \end{bmatrix} X^{-1}$

for all k.

- eg. $A = \begin{bmatrix} 2 & -3 \\ 2 & -5 \end{bmatrix}$

$$p(\lambda) = \begin{vmatrix} 2-\lambda & -3 \\ 2 & -5-\lambda \end{vmatrix} = (2-\lambda)(-5-\lambda) - (-3) 2 = \lambda^2 + 3\lambda - 4 = 0$$

$\lambda_1 = 1$, $\lambda_2 = -4$

$\lambda_1 = 1$

$A - I = \begin{bmatrix} 1 & -3 \\ 2 & -6 \end{bmatrix} \to \begin{bmatrix} 1 & -3 \\ 0 & 0 \end{bmatrix}$ eigenvector $\bar{x}_1 = \begin{pmatrix} 3 \\ 1 \end{pmatrix}$

$\lambda_2 = -4$

$A + 4I = \begin{bmatrix} 6 & -3 \\ 2 & -1 \end{bmatrix} \to \begin{bmatrix} 2 & -1 \\ 0 & 0 \end{bmatrix}$ eigenvector $\bar{x}_2 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$

Let $X = (\bar{X}_1, \bar{X}_2) = \begin{bmatrix} 3 & 1 \\ 1 & 2 \end{bmatrix}$ and $D = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & -4 \end{bmatrix}$

Then $X^{-1} = \frac{1}{5}\begin{bmatrix} 2 & -1 \\ -1 & 3 \end{bmatrix}$

$$AX = X \cdot D \iff X^{-1} \cdot A \cdot X = D$$

$$\iff \frac{1}{5}\begin{bmatrix} 2 & -1 \\ -1 & 3 \end{bmatrix} \cdot \begin{bmatrix} 2 & -3 \\ 2 & -5 \end{bmatrix}\begin{bmatrix} 3 & 1 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & -4 \end{bmatrix}$$

e.g. Let $A = \begin{bmatrix} 3 & -1 & -2 \\ 2 & 0 & -2 \\ 2 & -1 & -1 \end{bmatrix}$. Find A^{10}.

Solution: $p(\lambda) = \begin{vmatrix} 3-\lambda & -1 & -2 \\ 2 & -\lambda & -2 \\ 2 & -1 & -1-\lambda \end{vmatrix}$

$$= (3-\lambda) \cdot \begin{vmatrix} -\lambda & -2 \\ -1 & -1-\lambda \end{vmatrix} + \begin{vmatrix} 2 & -2 \\ 2 & -1-\lambda \end{vmatrix} - 2\begin{vmatrix} 2 & -\lambda \\ 2 & -1 \end{vmatrix}$$

$$= (3-\lambda)(\lambda^2+\lambda-2) + (-2-2\lambda+4) - 2(-2+2\lambda)$$

$$= 3\lambda^2+3\lambda-6-\lambda^3-\lambda^2+2\lambda+2-2\lambda+4-4\lambda$$

$$= -\lambda^3+2\lambda^2-\lambda = -\lambda(\lambda^2-2\lambda+1) = -\lambda(\lambda-1)^2 = 0$$

$\lambda_1 = 0$, $\lambda_2 = \lambda_3 = 1$.

$\lambda_1 = 0$

$$A = \begin{bmatrix} 3 & -1 & -2 \\ 2 & 0 & -2 \\ 2 & -1 & -1 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & -1 & 0 \\ 2 & 0 & -2 \\ 2 & -1 & -1 \end{bmatrix} \to \begin{bmatrix} 1 & -1 & 0 \\ 0 & 2 & -2 \\ 0 & 1 & -1 \end{bmatrix}$$

$$\rightarrow \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\begin{cases} x_1 = x_3 \\ x_2 = x_3 \end{cases} \Rightarrow \text{eiginvector } v_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \text{ of } \lambda_1 = 0.$$

$\lambda_2 = \lambda_3 = 1$

$$A - I = \begin{bmatrix} 2 & -1 & -2 \\ 2 & -1 & -2 \\ 2 & -1 & -2 \end{bmatrix} \rightarrow \begin{bmatrix} 2 & -1 & -2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$x_1 = \frac{1}{2}x_2 + x_3$, x_2, x_3 (free). x_1 (lead).

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} \frac{1}{2}\alpha + \beta \\ \alpha \\ \beta \end{pmatrix} = \alpha \cdot \begin{pmatrix} \frac{1}{2} \\ 1 \\ 0 \end{pmatrix} + \beta \cdot \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$$

There are two linearly independent vectors corresponding to 1.

$$v_2 = \begin{pmatrix} \frac{1}{2} \\ 1 \\ 0 \end{pmatrix}, \quad v_3 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$$

Let $X = (v_1, v_2, v_3) = \begin{bmatrix} 1 & \frac{1}{2} & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$ and $D = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$. Then.

$$X^{-1} A X = D \Rightarrow A = X D X^{-1}$$

$$A^{10} = X \cdot D^{10} \cdot X^{-1} = X \cdot D \cdot X^{-1} \quad \text{since } D^{10} = \begin{bmatrix} 0^{10} & & \\ & 1^{10} & \\ & & 1^{10} \end{bmatrix} = D$$

$$= A.$$

Remark. (1) X is not unique. e.g. One can take $v_2 = \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}$

and $X = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 0 \\ 1 & 0 & 1 \end{bmatrix}$

(2) For any k, $D^k = \begin{bmatrix} 0^k & & \\ & 1^k & \\ & & 1^k \end{bmatrix} = \begin{bmatrix} 0 & & \\ & 1 & \\ & & 1 \end{bmatrix} = D$

Therefore, $A^k = X \cdot D^k \cdot X^{-1} = X \cdot D \cdot X^{-1} = A$.

Remark. Not all matrices can be diagonalized

e.g. $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ has two eigenvalues $\lambda_1 = \lambda_2 = 1$

$A - I = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \Rightarrow A$ has only one eigenvector $v_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$.

A cannot be diagonalized.

Remark. $A^k = X \cdot D^k X^{-1}$ can be used to define the exponential of a matrix, where $X^{-1} A X = D = \begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{bmatrix}$

$$\boxed{e^A := X \cdot e^D X^{-1} = X \cdot \begin{bmatrix} e^{\lambda_1} & & \\ & \ddots & \\ & & e^{\lambda_n} \end{bmatrix} \cdot X^{-1}}$$

We used the exponential formula for a scalar a.

$$e^a = \sum_{k=0}^{\infty} \frac{1}{k!} a^k = 1 + a + \frac{1}{2} a^2 + \frac{1}{3!} a^3 + \cdots + \frac{1}{k!} a^k + \cdots$$

HW. 1. (a). (c). (e). 2. (a). 5. 9. 4. (a). 19. 31 (a).