Sec 5.6 The Gram-Schmit Orthogonalization Process

Gram-Schmit Process: Given n linearly independent vector $\overline{X}_1, \dots, \overline{X}_n$ in $\mathbb{R}^n$,

one can find an orthonormal basis of $\mathbb{R}^n$ using $\overline{X}_1, \dots, \overline{X}_n$

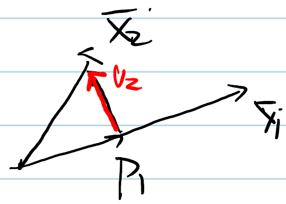
Step 1: Let $u_1 = \frac{1}{\|\overline{X}_1\|} \cdot \overline{X}_1$

Step 2: Find the projection of $\overline{X}_2$ onto u_1, namely P_1:

$$P_1 = \left(\frac{\langle \overline{X}_2, u_1 \rangle}{\|u_1\|^2} \cdot u_1 \right) = \langle \overline{X}_2, u_1 \rangle \cdot u_1$$

Let $V_2 = \overline{X}_2 - P_2$

(Then V_2 is orthogonal to u_1)

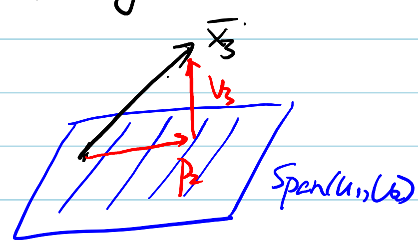


Let $u_2 = \frac{1}{\|V_2\|} \cdot V_2$

(Then $\{u_1, u_2\}$ is an orthonormal set)

Step 3: Find the projection of $\overline{X}_3$ onto $\mathrm{Span}(u_1, u_2)$, namely, P_2:

$$P_2 = \langle X_3, u_1 \rangle \cdot u_1 + \langle X_3, u_2 \rangle \cdot u_2$$



Let $V_3 = X_3 - P_2$

and $u_3 = \frac{1}{\|V_3\|} V_3$

(Then u_1, u_2, u_3 is an orthonormal set)

Recursively, after we have an orthonormal set $u_1, \dots, u_k$

Step (k+1)

Projection of $\overline{X}_{k+1}$ onto $\mathrm{Span}(u_1, \dots, u_k)$ is given by

$$P_k = \langle \overline{X}_{k+1}, u_1 \rangle u_1 + \langle \overline{X}_{k+1}, u_2 \rangle u_2 + \cdots + \langle \overline{X}_{k+1}, u_k \rangle \cdot u_k$$

Let $V_{k+1} = \overline{X}_{k+1} - P_{k+1}$ and $u_{k+1} = \frac{1}{\|V_{k+1}\|} \cdot V_{k+1}$

Stop at Step n, when we reach $\overline{X}_n$ and have

an orthonormal set $u_1, u_2, \dots, u_n$.

Remark. A key observation in G-S process is:

If p_k is the orthogonal projection of $\overline{x}_{k+1}$ onto $\mathrm{Span}(u_1,\cdots,u_k)$ then $\langle \overline{x}_{k+1}, u_j\rangle = \langle p_k, u_j\rangle$ for $j=1,\cdots,k$.

eg. Given $\overline{x}_1=(1,1,1)^T$, $\overline{x}_2=(1,0,1)^T$, $\overline{x}_3=(0,2,-1)^T$, use the G-S process to obtain an orthonormal basis for $\mathbb{R}^3$.

$$u_1=\frac{1}{\|\overline{x}_1\|}\cdot\overline{x}_1=\boxed{\frac{1}{\sqrt{3}}(1,1,1)^T}$$

$$\langle\overline{x}_2,u_1\rangle=(1,0,1)\cdot\frac{1}{\sqrt{3}}\begin{pmatrix}1\\1\\1\end{pmatrix}=\frac{1}{\sqrt{3}}(1+0+1)=\frac{2}{\sqrt{3}}.$$

$$p_1=\langle\overline{x}_2,u_1\rangle u_1=\frac{2}{\sqrt{3}}\frac{1}{\sqrt{3}}(1,1,1)^T=\frac{2}{3}(1,1,1)^T$$

$$v_2=\overline{x}_2-p_1=(1,0,1)^T-\left(\frac{2}{3},\frac{2}{3},\frac{2}{3}\right)^T=\left(\frac{1}{3},-\frac{2}{3},\frac{1}{3}\right)^T$$

$$\|v_2\|=\sqrt{\left(\frac{1}{3}\right)^2+\left(-\frac{2}{3}\right)^2+\left(\frac{1}{3}\right)^2}=\sqrt{\frac{1+4+1}{9}}=\sqrt{\frac{2}{3}}$$

$$u_2=\frac{1}{\|v_2\|}\cdot v_2=\frac{1}{\sqrt{\frac{2}{3}}}\left(\frac{1}{3},-\frac{2}{3},\frac{1}{3}\right)^T=\sqrt{\frac{3}{2}}\cdot\frac{1}{3}(1,-2,1)^T=\boxed{\frac{1}{\sqrt{6}}(1,-2,1)^T}$$

$$p_2=\langle\overline{x}_3,u_1\rangle\cdot u_1+\langle\overline{x}_3,u_2\rangle\cdot u_2$$

$$=(0,2,-1)\cdot\frac{1}{\sqrt{3}}\begin{pmatrix}1\\1\\1\end{pmatrix}\cdot\frac{1}{\sqrt{3}}(1,1,1)^T+(0,2,-1)\cdot\frac{1}{\sqrt{6}}(1,-2,1)^T\cdot\frac{1}{\sqrt{6}}(1,-2,1)^T$$

$$=\frac{1}{\sqrt{3}}(2-1)\cdot\frac{1}{\sqrt{3}}(1,1,1)^T+\frac{1}{\sqrt{6}}(-4-1)\cdot\frac{1}{\sqrt{6}}(1,-2,1)^T$$

$$=\frac{1}{3}(1,1,1)^T+\frac{-5}{6}(1,-2,1)^T$$

$$=\left(\frac{1}{3}-\frac{5}{6},\ \frac{1}{3}+\frac{5}{3},\ \frac{1}{3}-\frac{5}{6}\right)^T=\left(-\frac{1}{2},2,-\frac{1}{2}\right)^T$$

$$v_3=\overline{x}_3-p_2=(0,2,-1)^T-\left(-\frac{1}{2},2,-\frac{1}{2}\right)^T=\left(\frac{1}{2},0,-\frac{1}{2}\right)^T$$

$$\|v_3\|=\sqrt{\frac{1}{4}+\frac{1}{4}}=\frac{1}{\sqrt{2}},\quad \boxed{u_3=\frac{1}{\|v_3\|}\cdot v_3=\sqrt{2}\left(\frac{1}{2},0,-\frac{1}{2}\right)^T}$$

eg. Let $\bar{x}_1=(2,1,1,0)^T$, $\bar{x}_2=(1,-1,0,1)^T$

Use G-S process to find an orthonormal basis for $\mathrm{Span}(\bar{x}_1,\bar{x}_2)$

$$u_1=\frac{1}{\|\bar{x}_1\|}\cdot\bar{x}_1=\boxed{\frac{1}{\sqrt{6}}(2,1,1,0)^T},$$

$$\langle \bar{x}_2,u_1\rangle=(1,-1,0,1)\cdot\frac{1}{\sqrt{6}}(2,1,1,0)^T=\frac{1}{\sqrt{6}}(2-1+0+0)=\frac{1}{\sqrt{6}}$$

$$p_1=\langle \bar{x}_2,u_1\rangle\cdot u_1=\frac{1}{\sqrt{6}}\cdot\frac{1}{\sqrt{6}}(2,1,1,0)^T=\frac{1}{6}(2,1,1,0)^T$$

$$v_2=\bar{x}_2-p_1=(1,-1,0,1)^T-\frac{1}{6}(2,1,1,0)^T=\left(\frac{2}{3},-\frac{7}{6},-\frac{1}{6},1\right)^T$$

$$\|v_2\|=\sqrt{\left(\frac{2}{3}\right)^2+\left(\frac{-7}{6}\right)^2+\left(\frac{1}{6}\right)^2+1}=\sqrt{\frac{16+49+1+36}{36}}=\sqrt{\frac{102}{36}}=\sqrt{\frac{17}{6}}.$$

$$u_2=\frac{1}{\|v_2\|}\cdot v_2=\boxed{\sqrt{\frac{6}{17}}\cdot\left(\frac{2}{3},-\frac{7}{6},-\frac{1}{6},1\right)^T}$$

$\{u_1,u_2\}$ is an orthonormal basis for $\mathrm{Span}(\bar{x}_1,\bar{x}_2)$.

eg. (Application to the diagonalization of the symmetric matrix)

$$A=\begin{bmatrix}2&1\\1&2\end{bmatrix}$$

eigenvalue problem of A: $p(\lambda)=\det(\lambda I-A)-\quad -1=\lambda^2-4\lambda+3=0$

two eigenvalues: $\lambda_1=1$, $\lambda_2=3$

eigenvector: $\lambda_1=1$. $A-I=\begin{bmatrix}1&1\\1&1\end{bmatrix}\to\begin{bmatrix}1&1\\0&0\end{bmatrix}$ eigenvector: $v_1=\begin{pmatrix}1\\-1\end{pmatrix}$

$\lambda_2=3$. $A-3I=\begin{bmatrix}-1&1\\1&-1\end{bmatrix}\to\begin{bmatrix}-1&1\\0&0\end{bmatrix}$ eigenvector $v_2=\begin{pmatrix}1\\1\end{pmatrix}$.

$v_1\perp v_2$ since $\langle v_1,v_2\rangle=0$.

Let $u_1=\frac{1}{\|v_1\|}v_1=\frac{1}{\sqrt{2}}(1,-1)^T$, $u_2=\frac{1}{\|v_2\|}v_2=\frac{1}{\sqrt{2}}(1,1)^T$,

Then u_1,u_2 is an orthonormal basis and $Au_1=\lambda u_1$, $Au_2=\lambda u_2$.

Let $Q=(u_1,u_2)$. Then

$$A(u_1,u_2)=(u_1,u_2)\cdot\begin{bmatrix}1&0\\0&3\end{bmatrix}\iff AQ=Q\cdot D,\ \text{where } D=\begin{bmatrix}1&0\\0&3\end{bmatrix}.$$

Therefore, $Q^{-1} \cdot A \cdot Q = D$ (is as the usual diagonalization)

Notice that Q is an orthogonal matrix, we have $Q^{-1} = Q^T$.

Therefore, $Q^T \cdot A \cdot Q = D$.

Remark. In general, one can prove that if A is (an $n \times n$) symmetric matrix then one can find an orthogonal matrix Q such that

$$Q^T A Q = D, \text{ where } D \text{ is diagonal.}$$