Sec 6.5 Orthonormal Sets

Def. Let $V_1, \dots, V_n$ be nonzero vectors in $\mathbb{R}^n$

If $\langle V_i, V_j \rangle = 0$ for all $i \neq j$,

then $\{V_1, \dots, V_m\}$ is said to be an orthogonal set in $\mathbb{R}^n$

eg. $\{(1,1,1)^T, (2,1,-3)^T, (4,-5,1)^T\}$ is an orthogonal set in $\mathbb{R}^3$.

$$(1,1,1)\cdot\begin{pmatrix}2\\1\\-3\end{pmatrix} = 2+1-3=0$$

$$(1,1,1)\cdot\begin{pmatrix}4\\-5\\1\end{pmatrix} = 4-5+1=0$$

$$(2,1,-3)\cdot\begin{pmatrix}4\\-5\\1\end{pmatrix} = 8-5-3=0.$$

Theorem. If $\{V_1, \dots, V_n\}$ is an orthogonal set in $\mathbb{R}^m$, then $V_1, \dots, V_n$ are linearly independent.

Pf. of $n=3$: Suppose $c_1 \cdot V_1 + c_2 \cdot V_2 + c_3 \cdot V_3 = \vec{0}$

$$\begin{aligned}
0 &= \langle V_1, c_1 \cdot V_1 + c_2 \cdot V_2 + c_3 \cdot V_3 \rangle \\
&= \langle V_1, c_1 V_1 \rangle + \langle V_1, c_2 V_2 \rangle + \langle V_1, c_3 \cdot V_3 \rangle \\
&= c_1 \langle V_1, V_1 \rangle + c_2 \langle V_1, V_2 \rangle + c_3 \cdot \langle V_1, V_3 \rangle \\
&= c_1 \langle V_1, V_1 \rangle + 0 + 0
\end{aligned}$$

$v_1 \neq \vec{0} \Rightarrow \langle v_1, v_1 \rangle \neq 0 \Rightarrow c_1 = 0$

Similarly, $c_2 = c_3 = 0$.

Therefore, v_1, v_2, v_3 are linearly independent.

Def. An orthogonal set $\{v_1, \dots, v_n\}$ is called an orthonormal set if $\|v_j\| = 1$ for all $j = 1, \dots, n$.

- In the first example,

$v_1 = (1, 1, 1)^T$, $v_2 = (2, 1, -3)^T$, $v_3 = (4, -5, 1)^T$

is an orthogonal set.

Let $u_1 = \frac{1}{\|v_1\|} v_1 = \frac{1}{\sqrt{3}}(1, 1, 1)^T$

$u_2 = \frac{1}{\|v_2\|} \cdot v_2 = \frac{1}{\sqrt{14}}(2, 1, -3)^T$

$u_3 = \frac{1}{\|v_3\|} \cdot v_3 = \frac{1}{\sqrt{42}} \cdot (4, -5, 1)^T$

Then $\|u_1\| = \|u_2\| = \|u_3\| = 1$ and

$\{u_1, u_2, u_3\}$ is an orthonormal set

Theorem. Let $\{u_1, u_2, \cdots, u_n\}$ be an orthonormal set in $\mathbb{R}^n$.
Then $\{u_1, \cdots, u_n\}$ is a basis for $\mathbb{R}^n$.
$\{u_1, u_2, \ldots, u_n\}$ is called an orthonormal basis.

e.g. $u_1 = \begin{pmatrix} \frac{1}{2} \\ \frac{\sqrt{3}}{2} \end{pmatrix}$, $u_2 = \begin{pmatrix} -\frac{\sqrt{3}}{2} \\ \frac{1}{2} \end{pmatrix}$ is an orthonormal basis for $\mathbb{R}^2$.

since $\langle u_1, u_2 \rangle = \frac{1}{2} \cdot \left(-\frac{\sqrt{3}}{2}\right) + \frac{\sqrt{3}}{2} \cdot \frac{1}{2} = 0$.
$(u_1 \perp u_2)$

and $\|u_1\| = \sqrt{\frac{1}{4} + \frac{3}{4}} = 1$.
$\|u_2\| = \sqrt{\frac{3}{4} + \frac{1}{4}} = 1$.

Theorem. Let $\{u_1, \ldots, u_n\}$ be an orthogonal basis for $\mathbb{R}^n$.
If $v = \sum_{i=1}^{n} c_i u_i$, then
$$c_i = \langle v, u_i \rangle.$$
If $w = \sum_{i=1}^{n} a_i \cdot u_i$, then
$$\langle v, w \rangle = \sum_{i=1}^{n} c_i \cdot a_i$$
In particular,
$$\|v\|^2 = \sum_{i=1}^{n} a_i^2 \quad \text{(Parseval's Formula)}$$

9. $u_1 = \frac{1}{\sqrt{3}}(1,1,1)^T$

$u_2 = \frac{1}{\sqrt{14}}(2,1,-3)^T$

$u_3 = \frac{1}{\sqrt{42}} \cdot (4,-5,1)^T.$

Let $\overline{x} = (1,0,-2)^T$. Find c_1, c_2, c_3 such that

$$\overline{x} = c_1 \cdot u_1 + c_2 \cdot u_2 + c_3 \cdot u_3$$

(Write $\overline{x}$ as a linear combination of u_1, u_2, u_3).

Because u_1, u_2, u_3 are orthonormal,

$$c_1 = \langle \overline{x}, u_1 \rangle = (1,0,-2) \cdot \frac{1}{\sqrt{3}} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \frac{1}{\sqrt{3}}(1+0-2) = \frac{-1}{\sqrt{3}}.$$

$$c_2 = \langle \overline{x}, u_2 \rangle = (1,0,-2) \frac{1}{\sqrt{14}} \begin{pmatrix} 2 \\ 1 \\ -3 \end{pmatrix} = \frac{1}{\sqrt{14}}(2+0+6) = \frac{1}{\sqrt{14}} \cdot 8$$

$$c_3 = \langle \overline{x}, u_3 \rangle = (1,0,-2) \frac{1}{\sqrt{42}} \cdot \begin{pmatrix} 4 \\ -5 \\ 1 \end{pmatrix} = \frac{1}{\sqrt{42}} \cdot (4+0-2) = \frac{1}{\sqrt{42}} \cdot 2.$$

One can check that

$\|\overline{x}\|^2 = 1^2 + 2^2 = 5$

$$c_1^2 + c_2^2 + c_3^2 = \left(\frac{-1}{\sqrt{3}}\right)^2 + \left(\frac{8}{\sqrt{14}}\right)^2 + \left(\frac{2}{\sqrt{42}}\right)^2 = \frac{1}{3} + \frac{64}{14} + \frac{4}{42} = 5 = \|\overline{x}\|^2$$

Def. An $n\times n$ matrix Q is said to be an orthogonal matrix if the column vectors of Q form an orthonormal set in $\mathbb{R}^n$.

Thm Q is orthogonal if and only if $Q^T \cdot Q = I$.

eg $Q = \begin{bmatrix} \frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{bmatrix}$ is an orthogonal matrix since

$u_1 = \begin{pmatrix} \frac{1}{2} \\ \frac{\sqrt{3}}{2} \end{pmatrix}$, $u_2 = \begin{pmatrix} -\frac{\sqrt{3}}{2} \\ \frac{1}{2} \end{pmatrix}$ is an orthonormal set in $\mathbb{R}^2$.

And,

$$Q^T \cdot Q = \begin{bmatrix} \frac{1}{2} & \frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & \frac{1}{2} \end{bmatrix} \cdot \begin{bmatrix} \frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Prop. If Q is an orthogonal matrix, then

(1). $Q^T Q = I$.

(2) $Q^T = Q^{-1}$

(3). $\langle Q\bar{x}, Q\bar{y} \rangle = \langle \bar{x}, \bar{y} \rangle$

(4). $\| Q\bar{x} \|_2 = \| \bar{x} \|_2$