Sec 4.2 Matrix representations of Linear Transformations.

- Each $m\times n$ matrix A defines a linear transformation

  $L: \mathbb{R}^n \to \mathbb{R}^m$ as

  $L(\bar{x}) = A\cdot\bar{x}$ for $\bar{x} \in \mathbb{R}^n$

  We denote this map by L_A sometimes to indicate it corresponds to the matrix A.

Theorem. For any linear transformation $L: \mathbb{R}^n \to \mathbb{R}^m$, there exists an $m\times n$ matrix A such that

$$L(\bar{x}) = A\bar{x}, \quad \bar{x} \in \mathbb{R}^n.$$

A is called the (standard) matrix representation of L and

$$A = \left(L(\bar{e}_1), L(\bar{e}_2), \cdots, L(\bar{e}_n) \right),$$ where

$\{\bar{e}_1, \cdots, \bar{e}_n\}$ is the standard basis of $\mathbb{R}^n$.

eg. Let $L: \mathbb{R}^3 \to \mathbb{R}^2$ be given as

$$L(\bar{x}) = (x_1 + x_2,\ x_2 + x_3)^T \quad \text{for } \bar{x} = (x_1, x_2, x_3)^T \in \mathbb{R}^3.$$

Find the matrix representation of L.

$$L\begin{pmatrix}1\\0\\0\end{pmatrix} = \begin{pmatrix}1\\0\end{pmatrix},\ L\begin{pmatrix}0\\1\\0\end{pmatrix} = \begin{pmatrix}1\\1\end{pmatrix},\ L\begin{pmatrix}0\\0\\1\end{pmatrix} = \begin{pmatrix}0\\1\end{pmatrix},\ A = \begin{bmatrix}1 & 1 & 0\\ 0 & 1 & 1\end{bmatrix}$$

eg. Let $L: \mathbb{R}^2 \to \mathbb{R}^2$ be given by

$$L\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} \frac{1}{2}x_1 - \frac{\sqrt{3}}{2}x_2 \\ \frac{\sqrt{3}}{2}x_1 + \frac{1}{2}x_2 \end{pmatrix} = \begin{bmatrix} \frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

The matrix representation is $A = \begin{bmatrix} \frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{bmatrix}$

Notice that $A = \begin{bmatrix} \cos\frac{\pi}{3} & -\sin\frac{\pi}{3} \\ \sin\frac{\pi}{3} & \cos\frac{\pi}{3} \end{bmatrix}$

The linear transformation L rotates each vector $\bar{x} \in \mathbb{R}^2$ by the angle $\frac{\pi}{3}$ counterclockwise

- Matrix representation with respecto to ordered bases.

Let $L: V \to W$ be a linear transformation

V has an ordered basis $\{v_1, \cdots, v_n\}$
W has an ordered basis $\{w_1, \cdots, w_m\}$

$\bar{a}_j = [L(v_j)]_{\{w_1, \cdots, w_m\}} \in \mathbb{R}^m$ be the coordinates vector of $L(v_j)$ in W (with respect to $w_1, \cdots, w_m$) for $j = 1, 2, \cdots, n$

$A = (\bar{a}_1, \bar{a}_2, \cdots, \bar{a}_n)$ is called the matrix representation of L with respect to these two bases.

e.g. Let $L: P_3 \to \mathbb{R}^3$ be defined by

$$L(a_0 + a_1 x + a_2 x^2) = (a_0, a_1, a_2)^T$$

Find the matrix representation of L with respect to the standard basis $\{1, x, x^2\}$ of P_3 and $\{\bar{e}_1, \bar{e}_2, \bar{e}_3\}$ of $\mathbb{R}^3$

$$L(1) = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad L(x) = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \quad L(x^2) = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

The matrix representation of L is

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

e.g. (HW 14*).

Let $L: P_3 \to P_2$ be defined by

$$L(p(x)) = p'(x) + p(0) \quad \text{for } p(x) \in P_3.$$

Consider the ordered bases $\{x^2, x, 1\}$ of P_3 and $\{2, 1-x\}$ of P_2.

Find the matrix representation with respect to these bases.

Solution: $L(x^2) = 2x = 2 - 2(1-x)$

The coordinates of $L(x^2)$ are $\begin{pmatrix} 1 \\ -2 \end{pmatrix}$ w.r.t. $\{2, 1-x\}$.

$L(x) = 1 = \frac{1}{2}\cdot 2 + 0\cdot(1-x)$

The coordinates are $\begin{pmatrix} \frac{1}{2} \\ 0 \end{pmatrix}$

$L(1) = 1 = \frac{1}{2}\cdot 2 + 0\cdot(1-x)$

The coordinates are $\begin{pmatrix} \frac{1}{2} \\ 0 \end{pmatrix}$.

The matrix representation is

$$A = \begin{bmatrix} 1 & \frac{1}{2} & \frac{1}{2} \\ -2 & 0 & 0 \end{bmatrix}$$

$p(x) = x^2 + 2x - 3$ has coordinates $\begin{bmatrix} 1 \\ 2 \\ -3 \end{bmatrix}$ w.r.t. $\{x^2, x, 1\}$

Then $L(p(x))$ has coordinates.

$$A \cdot \begin{bmatrix} 1 \\ 2 \\ -3 \end{bmatrix} = \begin{bmatrix} 1 & \frac{1}{2} & \frac{1}{2} \\ -2 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ -3 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} \\ -2 \end{bmatrix}$$