Chapter 4. Linear Transformation

Sec 4.1 Definition and Examples

Def. A mapping L from a vector space V into a vector space W is said to be a $\boxed{\text{linear transformation}}$ if

$$(1)\quad L(v_1+v_2)=L(v_1)+L(v_2)$$

$$(2)\quad L(\alpha\cdot v_1)=\alpha\cdot L(v_1)$$

for all vectors $v_1, v_2 \in V$ and scalar α.

Remark. • We use the notation $L: V \to W$ to indicate that a mapping L maps elements v in V to $L(v)$ in W.

• (1) and (2) hold if and only if

$$L(\alpha v_1+\beta v_2)=\alpha\cdot L(v_1)+\beta\cdot L(v_2)$$

for all $v_1, v_2 \in V$ and scalars α, β.

• A linear transformation L is also called a linear map / linear operator / linear function, etc.

• Linear maps from $\mathbb{R}^n$ to $\mathbb{R}^m$

e.g. Linear function from $\mathbb{R}$ to $\mathbb{R}$. $f(x)=3x$.

e.g. $\mathbb{R}^2 \to \mathbb{R}^2$: For $\bar{x}=\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$, $L(\bar{x})=3\bar{x}$

eg $f(x_1, x_2) = (x_1^2, x_2^2)$ is NOT a linear map from $\mathbb{R}^2$ to $\mathbb{R}^2$.

$M(x_1, x_2) = \sqrt{x_1^2 + x_2^2}$ is NOT a linear map from $\mathbb{R}^2$ to $\mathbb{R}^1$.

eg Let $A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \\ 1 & 1 \end{bmatrix}$. For $\bar{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$, let

$$L(\bar{x}) = A\bar{x} = \begin{bmatrix} x_2 \\ x_1 \\ x_1 + x_2 \end{bmatrix}.$$ L is a linear map from $\mathbb{R}^2$ to $\mathbb{R}^3$

since $L(\bar{x} + \bar{y}) = A(\bar{x} + \bar{y}) = A\bar{x} + A\bar{y} = L(\bar{x}) + L(\bar{y})$

and $L(\alpha\bar{x}) = A(\alpha\bar{x}) = \alpha(A\bar{x}) = \alpha \cdot L(\bar{x})$

for any $\bar{x}, \bar{y} \in \mathbb{R}^2$ and scalar α.

Theorem. Let $L: V \to W$ be a linear map from vector space V to vector space W. Let 0_V and 0_W be the corresponding zero vectors in V and W.

Then. $L(0_V) = 0_W$.

eg For a 2×2 matrix A and a nonzero vector $\bar{b} \in \mathbb{R}^2$,

$L(\bar{x}) = A\bar{x} + \bar{b}$ is Not a linear map since

$L(\bar{0}) = \bar{b} \neq \bar{0}$

Other examples.

- Differential operator $L = \frac{d}{dx}$ from P_{n+1} to P_n.

  Let P_3 and P_2 be the usual polynomial vector space.

  For $p(x) = a_0 + a_1 X + a_2 X^2 \in P_3$, let

  $$L(p) = \frac{d\,p(x)}{dx} = p'(x) = a_1 + 2a_2 X$$

  Then $L: P_3 \to P_2$ is a linear map.

- Integral operator L from P_n to P_{n+1}

  For $q(x) = a_0 + a_1 X \in P_2$, let

  $$L(q) = \int_0^x q(t)\,dt = a_0 X + \frac{1}{2} a_1 X^2$$

  Then L is a linear map from P_2 to P_3

- Linear map from P_n to $\mathbb{R}^n$ and from $\mathbb{R}^n$ to P_n

  For $p(x) = a_0 + a_1 X + a_2 X^2 \in P_3$, let

  $$L_1(p) = (a_0, a_1, a_2)^T \in \mathbb{R}^3$$

  Then $L_1: P_3 \to \mathbb{R}^3$ is a linear map.

  For $\bar{a} = (a_0, a_1, a_2)^T \in \mathbb{R}^3$, let

  $$L_2(\bar{a}) = a_0 + a_1 X + a_2 X^2 \in P_3$$

  Then $L_2: \mathbb{R}^3 \to P_3$ is a linear map

The image and kernel.

Def. Let $L: V \to W$ be a linear transformation. The kernel of L, denoted $\ker(L)$, is

$$\ker(L) = \{ v \in L \mid L(v) = 0_W \}.$$

Remark. The concepts "kernel" and "null space" are exactly the same

Def. Let $L: V \to W$ be a linear transformation and let S be a subspace of V.

$L(S) = \{ f(v) \mid v \in S \}$ is called the image of S.

$L(V)$, the image of the entire vector space, is called the range of L.

eg. Let $L: P_3 \to P_2$ be the differential operator defined by

$$L(p) = p'(x) \quad \text{for } p(x) \in P_3.$$

$L(p) = 0$ implies $p'(x) = 0 \Rightarrow p(x) = \text{constant}$

i.e. $\ker(L) = \{ a_0 \mid a_0 \in \mathbb{R} \}$ (collection of all constant functions)

eg. Let $L: \mathbb{R}^2 \to \mathbb{R}^2$ be defined by

$$L\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} x_1 \\ 0 \end{pmatrix}$$

The range of L, $L(\mathbb{R}^2)$ is $\{ \alpha \bar{e}_1 \mid \alpha \in \mathbb{R} \}$ which is the one dimensional subspace of $\mathbb{R}^2$ spanned by $\bar{e}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$.