Sec 3.4 Basis and dimension

Def. The vectors $v_1, v_2, \dots, v_n$ form a basis for a vector space V if and only if

(i). $v_1, \dots, v_n$ are linearly independent

(ii) $v_1, \dots, v_n$ span V.

e.g. In $\mathbb{R}^n$,

$$\bar{e}_1 = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \quad \bar{e}_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \quad \dots, \quad \bar{e}_n = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix}$$ form a

$\boxed{\text{standard basis}}$ for any n.

- Given V, one can have many bases (The basis is not unique)

e.g. In $\mathbb{R}^3$,

$$\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}$$ also form a basis.

since (i) they are linearly independent.

(ii). they span $\mathbb{R}^3$.

Theorem. Any basis in $\mathbb{R}^n$ must contain exactly n vectors.

Theorem. Any collection of more than n vectors in $\mathbb{R}^n$ is linearly dependent.

Remark. The general version for abstract vector space V is Theorem 3.4.1 in the textbook.

Def. Let V be a vector space.
If V has a basis consisting of n vectors, we say V has dimension n.

Remark. If V only contains the zero vector $\bar{0}$, then V has no basis, and has dimension 0.
If V has a basis consisting of infinitely many vectors, we say V is infinite dimensional.

e.g. Let P be the vector space of all polynomials (any degree).
$\{1, x, x^2, \cdots, x^n, \cdots\}$ is a basis of P, which contains infinitely many vectors.
P is infinite dimensional.

Remark. We will mainly focus on finite dimensional space in this course.

- We can also talk about the basis and the dimension of a vector subspace.

Question: Given V, and any n vectors $V_1, \dots, V_n$,

$\mathrm{Span}(V_1, \dots, V_n)$ is a subspace of V.

Is $\{V_1, \dots, V_n\}$ a basis of $\mathrm{Span}(V_1, \dots, V_n)$?

Is the dimension of $\mathrm{Span}(V_1, \dots, V_n)$ n?

Answer: No.

$\{V_1, \dots, V_n\}$ forms a basis for $\mathrm{Span}(V_1, \dots, V_n)$ and dimension of $\mathrm{Span}(V_1, \dots, V_n) = n$ if and only if $\{V_1, \dots, V_n\}$ are linearly independent.

eg. In $\mathbb{R}^3$, let $V_1 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$, $V_2 = \begin{pmatrix} -1 \\ -1 \\ 0 \end{pmatrix}$, $V_3 = \begin{pmatrix} 2 \\ 2 \\ 0 \end{pmatrix}$

then $\mathrm{Span}(V_1, V_2, V_3) = \{ \alpha_1 \cdot V_1 + \alpha_2 V_2 + \alpha_3 \cdot V_3 \}$

$$= \left\{ \alpha \cdot \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \right\}$$

Actually, any single one of V_1, V_2, V_3 is a basis of $\mathrm{Span}(V_1, V_2, V_3)$. The dimension of $\mathrm{Span}(V_1, V_2, V_3)$ is 1.

e.g. Let P_3 be the vector space of all polynomials of degree less than 3.

Let $p_0(x)=1$, $p_1(x)=x$, $p_2(x)=x^2$

Then $\{p_0, p_1, p_2\}$ forms a basis of P_3.

The dimension of P_3 is 3.

Consider $q_0(x)=1+x$, $q_1(x)=x^2$, $q_2(x)=1+x+x^2$

$\text{Span}(q_0, q_1, q_2)$ is a subspace of P_3.

Question: Is $\{q_0, q_1, q_2\}$ a basis for $\text{Span}(q_0, q_1, q_2)$?

What's the dimension of $\text{Span}\{q_0, q_1, q_2\}$?

Answer: $\{q_0, q_1, q_2\}$ is not a basis since

$q_0+q_1=q_2$. They are linearly dependent.

q_0, q_1 are linearly independent since

$c_0(1+x)+c_1\cdot x^2=0$ implies $c_0=c_1=0$.

$\{q_0, q_1\}$ actually forms a basis of $\text{Span}(q_0, q_1, q_2)$

and the dimension of $\text{Span}(q_0, q_1, q_2)$ is 2.