Sec 2.1. The determinant of a matrix.

Def. For each $n \times n$ matrix $A = (a_{ij})$, we associate a scalar, $\det(A)$ to A, which is called the determinant of A

We also use the following notations for $\det A$

$$\det A = \det \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \vdots & & & \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix} = \begin{vmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \vdots \\ a_{n1} & \cdots & a_{nn} \end{vmatrix}$$

(with two vertical lines).

Formulas of $\det A$ for a $n \times n$ A.

2×2: $$\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11} \cdot a_{22} - a_{12} \cdot a_{21}$$

e.g. $$\begin{vmatrix} 2 & 5 \\ 1 & 3 \end{vmatrix} = 2 \cdot 3 - 5 = 1, \quad \det \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = 0 \cdot 0 - 1 = -1$$

$\boxed{3\times 3:}$

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

$$= a_{11} \underbrace{\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix}}_{\det(M_{11})} - a_{12} \cdot \underbrace{\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix}}_{\det(M_{12})} + a_{13} \cdot \underbrace{\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}}_{\det(M_{13})}.$$

For a $n\times n$ matrix $A = (a_{ij})$, we define M_{ij} to be the sub-matrix of A by deleting the i-th row and j-th column of A, i.e.,

$$M_{ij} = \begin{bmatrix} a_{11} & \cdots & a_{1j} & \cdots & a_{1n} \\ \vdots & & & & \\ \cancel{a_{i1}} & \cancel{\cdots} & \cancel{a_{ij}} & \cancel{\cdots} & \cancel{a_{in}} \\ \vdots & & & & \\ a_{n1} & \cdots & a_{nj} & \cdots & a_{nn} \end{bmatrix},$$ which is a $(n-1)\times(n-1)$ matrix.

We call $\det(M_{ij})$ the minor of a_{ij}, and $A_{ij} = (-1)^{i+j}\det(M_{ij})$ the cofactor of a_{ij}.

Remark. In some textbooks, people denote $\det(M_{ij})$ by M_{ij} instead.

With these notations, we can rewrite

$$\det A = a_{11} \cdot A_{11} + a_{12} \cdot A_{12} + a_{13} \cdot A_{13}.$$

Prop. $\det A = a_{11} \cdot A_{11} + a_{21} \cdot A_{21} + a_{31} \cdot A_{31}$

Def. (The determinant for general $n \times n$ matrix A)

$$\det A = a_{11} A_{11} + a_{12} A_{12} + \cdots + a_{1n} A_{1n}$$

where $A_{1j} = (-1)^{1+j} \det(M_{1j})$ $j = 1, \ldots, n$ are the cofactors associated with the entries in the first row of A.

Thm 2.1.1.

$$\det A = a_{i1} A_{i1} + \cdots + a_{in} A_{in}$$ Expanding with respect to the i-th row

$$= a_{1j} A_{1j} + \cdots + a_{nj} A_{nj}$$ Expanding w.r.t. the j-th column.

for all $i = 1, \ldots, n$ and $j = 1, \ldots, n$

Thm 2.1.2 $\det A = \det(A^T)$

Thm 2.1.3 If $A = \begin{pmatrix} a_{11} & & * \\ & \ddots & \\ 0 & & a_{nn} \end{pmatrix}$ or $A = \begin{pmatrix} a_{11} & & 0 \\ * & \ddots & \\ \vdots & & \\ * & \cdots & a_{nn} \end{pmatrix}$

then $\det A = a_{11} \cdot a_{22} \cdots a_{nn}$.

Thm 2.1.4.

(i) If A has a row or column entirely zero, then $\det A = 0$

(ii) If A has two identical rows or columns, then $\det A = 0$.

Thm If $\det A \neq 0$, then A is non-singular

HW. 1 (a), (b). 2. (a), (b). 3. (a), (c), (e), (g). 5, 6, 8*, 13 e.g. Evaluate the following determinants:

(Exe §1.4. 3 (d))

$$\begin{vmatrix} 4 & 3 & 0 \\ 3 & 1 & 2 \\ 5 & -1 & -4 \end{vmatrix} = 4\begin{vmatrix} 1 & 2 \\ -1 & -4 \end{vmatrix} - 3\begin{vmatrix} 3 & 2 \\ 5 & -4 \end{vmatrix} + 0\cdot\begin{vmatrix} 3 & 1 \\ 5 & -1 \end{vmatrix}$$

$$= 4(-4-(-2)) - 3(-12-5\cdot 2) + 0$$

$$= 4(-2) - 3\cdot(-12-10) = -8+66 = 58.$$

(Exe §1.4 4.(b))

$$\begin{vmatrix} 2 & 0 & 0 \\ 4 & 1 & 0 \\ 7 & 3 & -2 \end{vmatrix} = 2\begin{vmatrix} 1 & 0 \\ 3 & -2 \end{vmatrix} - 0\cdot\begin{vmatrix} 4 & 0 \\ 7 & -2 \end{vmatrix} + 0\cdot\begin{vmatrix} 4 & 1 \\ 7 & 3 \end{vmatrix}$$

$$= 2\cdot(1\cdot(-2) - 0\cdot 3) = 2\cdot 1\cdot(-2) = -4.$$

e.g. (4×4 matrix)

$$\det A = \begin{vmatrix} 2 & -1 & 0 & 3 \\ 1 & 0 & -2 & 0 \\ 1 & -1 & 1 & 2 \\ 0 & -2 & 0 & 0 \end{vmatrix} = 2\cdot\begin{vmatrix} 0 & -2 & 0 \\ -1 & 1 & 2 \\ -2 & 0 & 0 \end{vmatrix} - (-1)\cdot\begin{vmatrix} 1 & -2 & 0 \\ 1 & 1 & 2 \\ 0 & 0 & 0 \end{vmatrix} + 0\cdot\begin{vmatrix} \cdots \\ \cdots \\ \cdots \end{vmatrix} - 3\cdot\begin{vmatrix} 1 & 0 & -2 \\ 1 & -1 & 1 \\ 0 & -2 & 0 \end{vmatrix}$$

$$\begin{vmatrix} 0 & -2 & 0 \\ -1 & 1 & 2 \\ -2 & 0 & 0 \end{vmatrix} = 0\begin{vmatrix} \cdots \end{vmatrix} + 2\cdot\begin{vmatrix} -1 & 2 \\ -2 & 0 \end{vmatrix} + 0\begin{vmatrix} \cdots \end{vmatrix} = 2(0+4) = 8$$

$$\begin{vmatrix} 1 & -2 & 0 \\ 1 & 1 & 2 \\ 0 & 0 & 0 \end{vmatrix} = 0, \quad \begin{vmatrix} 1 & 0 & -2 \\ 1 & -1 & 1 \\ 0 & -2 & 0 \end{vmatrix} = 1\cdot\begin{vmatrix} -1 & 1 \\ -2 & 0 \end{vmatrix} - 0\begin{vmatrix} \cdots \end{vmatrix} - 2\begin{vmatrix} 1 & -1 \\ 0 & -2 \end{vmatrix}$$

$$= 1\cdot(0+2) - 2(-2-0) = 6$$

$$\det A = 2\cdot 8 + 1\cdot 0 + 0 - 3\cdot 6 = -2$$

Remark. The last row only has one nonzero entry. The computation will be simpler if we expand the determinant w.r.t. the last row. $\det A = (-1)^{4+2}\cdot(-2)\cdot\begin{vmatrix} 2 & 0 & 3 \\ 1 & -2 & 0 \\ 1 & 1 & 2 \end{vmatrix}$