Chapter 1. Matrices and Systems of Equations

Lec 1. Jan. 7th, Mon.

Sec 1.1 Systems of linear equations.

Def. $m \times n$ linear system: m linear equations in n unknown variables.

m: number of equations. n: number of unknowns.

$$\text{m rows}\left\{\begin{array}{l} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2 \\ \vdots \qquad\qquad\qquad\qquad\qquad\qquad\qquad \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n = b_m \end{array}\right. \qquad \begin{array}{l} \text{1st eq} \\ \text{2nd eq} \\ \vdots \\ \text{m-th eq.} \end{array}$$

n columns.

$x_1, \dots, x_n$ are unknowns (to be solved for)

a_{ij}, $1 \le i \le m$, $1 \le j \le n$: coefficient.

b_i, $1 \le i \le m$: right-hand sides.

Goal: Given a_{ij}, b_i, solve for $x_1, \dots, x_n$.

Def. Solution set of the system: the collection of all the n-tuple $(x_1, \dots, x_n)$ that satisfies all the m-equations of the system.

e.g. 2×2 Systems.

(1) $\begin{cases} 2x_1 + x_2 = 3 \\ x_1 - x_2 = 0 \end{cases}$ has a unique solution $x_1=1, x_2=1$.

(2) $\begin{cases} 2x_1 + x_2 = 3 \\ 3x_1 \quad\quad = 3 \end{cases}$ has the same solution $x_1=1, x_2=1$.

(3) $\begin{cases} 2x_1 + x_2 = 3 \\ 4x_1 + 2x_2 = 6 \end{cases}$ has infinitely many solutions.

(4) $\begin{cases} 2x_1 + x_2 = 3 \\ 4x_1 + 2x_2 = 5 \end{cases}$ has no solutions.

- (1), (2), (3) are called consistent (The system has at least one solution)
- (4) is called inconsistent. (The system has NO solution).
- (1) and (2) are called Equivalent Systems since they have the same set of solution(s).
- One central goal of Linear algebra is to study different (non) equivalent linear systems and to find the solution set in a systematically way for any $m\times n$ systems.

$n\times n$ systems, triangular form, back substitution

Def. Strict triangular form (system): the k-th equation does not contain variables $x_1, x_2, \ldots, x_{k-1}$.

e.g.
$$\begin{cases} 3x_1 + 2x_2 + x_3 = 1 \\ x_2 - x_3 = 2 \\ 2x_3 = 4 \end{cases} \iff \begin{cases} 3x_1 + 2x_2 + x_3 = 1 \\ \qquad\quad x_2 - x_3 = 2 \\ \qquad\qquad\quad 2x_3 = 4 \end{cases}$$

(upper triangle)

Strict triangular system can be solved using $\boxed{\text{back substitution}}$

Start from the last equation (n-th), and solve for x_k from $k=n$ to $k=1$.

e.g. $2x_3 = 4 \Rightarrow x_3 = 2 \xrightarrow{\text{Plug into the 2nd eq.}} x_2 - 2 = 2$

$\Rightarrow x_2 = 4$

$\xrightarrow[\text{the 1st eq.}]{\text{Plug } x_3=2,\ x_2=4 \text{ into}} 3x_1 + 8 + 2 = 1$

$\Rightarrow x_1 = -3.$

The solution of the system is $(-3, 4, 2)$
(or $x_1 = -3,\ x_2 = 4,\ x_3 = 2$).

- In general, we will use the following operations to convert one system to an equivalent system, which is triangular.
  - Interchange two equations
  - Add (subtract) a multiple of one equation to (from) another.

$$\begin{array}{llll}
① & x_1 + 2x_2 + x_3 = 3 & & x_1 + 2x_2 + x_3 = 3 & & x_1 + 2x_2 + x_3 = 3 \\
② & 3x_1 - x_2 - 3x_3 = -1 & \boxed{② - 3\cdot①} & 0 - 7x_2 - 6x_3 = -10 & & -7x_2 - 6x_3 = -10 \\
③ & 2x_1 + 3x_2 + x_3 = 4 & \boxed{③ - 2\cdot①} & 0 - x_2 - x_3 = -2 & \boxed{③ - \frac{1}{7}②} & -\frac{1}{7}x_3 = -\frac{4}{7}
\end{array}$$

- From the above process, we see that it is sometimes enough to consider only the array of coefficients a_{ij} and b_i, omitting $x_1, \dots, x_n$.
- Coefficient matrix and augmented matrix.

Def: A $m \times n$ matrix is a rectangular array of numbers, with m rows, and n columns.

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$$ is the coefficient matrix of

a $m \times n$ linear system

$$\begin{cases} a_{11}x_1 + \cdots + a_{1n}x_n = b_1 \\ \vdots \\ a_{m1}x_1 + \cdots + a_{mn}x_n = b_m. \end{cases}$$

If we also include $\begin{pmatrix} b_1 \\ \vdots \\ b_m \end{pmatrix}$ as the $n+1$ column, we obtain the augmented matrix, which is $m \times (n+1)$.

$$\left[\begin{array}{ccc|c} a_{11} & \cdots & a_{1n} & b_1 \\ a_{21} & \cdots & a_{2n} & b_2 \\ \vdots & & & \vdots \\ a_{m1} & \cdots & a_{mn} & b_m \end{array}\right].$$

- Elementary Row Operations.

I. Interchange two rows

II. Multiply a row by a nonzero number

III. Replace a row by its sum with a multiple of another row.

$$\begin{array}{l} \text{pivotal row} \rightarrow \\ \\ \\ \end{array}\left[\begin{array}{ccc|c} 1 & 2 & 1 & 3 \\ 3 & -1 & -3 & -1 \\ 2 & 3 & 1 & 4 \end{array}\right] \longrightarrow \left[\begin{array}{ccc|c} 1 & 2 & 1 & 3 \\ 0 & -7 & -6 & -10 \\ 0 & -1 & -1 & -2 \end{array}\right]\begin{array}{l} \leftarrow \text{Fix 1st row, operate from 2nd} \\ \leftarrow \text{pivotal row.} \\ \\ \end{array}$$

$$\longrightarrow \left[\begin{array}{ccc|c} 1 & 2 & 1 & 3 \\ 0 & -7 & -6 & -10 \\ 0 & 0 & -\frac{1}{7} & -\frac{4}{7} \end{array}\right]$$

triangular form, stop.

The last augmented matrix correspond a new system, which is triangular.

$$x_1 + 2x_2 + x_3 = 3 \Rightarrow x_1 - 4 + 4 = 3 \overset{⑤}{\Rightarrow} x_1 = 3$$

$$-7x_2 - 6x_3 = -10 \Rightarrow -7x_2 - 24 = -10 \overset{③}{\Rightarrow} x_2 = -2$$

$$-\frac{1}{7}x_3 = -\frac{4}{7} \overset{①}{\Rightarrow} x_3 = 4$$

④ ②

The solution: $x_1 = 3$, $x_2 = -2$, $x_3 = 4$.

· Example 4 in textbook, Page 8.

$$\begin{aligned} -x_2 - x_3 + x_4 &= 0 \\ x_1 + x_2 + x_3 + x_4 &= 6 \\ 2x_1 + 4x_2 + x_3 - 2x_4 &= -1 \\ 3x_1 + x_2 - 2x_3 + 2x_4 &= 3 \end{aligned} \qquad \left[\begin{array}{cccc|c} 0 & -1 & -1 & 1 & 0 \\ 1 & 1 & 1 & 1 & 6 \\ 2 & 4 & 1 & -2 & -1 \\ 3 & 1 & -2 & 2 & 3 \end{array}\right]$$

Step 1: interchange row 1 and row 2

$$\longrightarrow \left[\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 6 \\ 0 & -1 & -1 & 1 & 0 \\ 2 & 4 & 1 & -2 & -1 \\ 3 & 1 & -2 & 2 & 3 \end{array}\right]$$

row 2 is done, it is enough to deal with row3, row4 only.

$$③ - 2① \rightarrow \quad 0 \quad 2 \quad -1 \quad -4 \quad | \quad -13$$

$$④ - 3① \rightarrow \quad 0 \quad -2 \quad -5 \quad -1 \quad | \quad -15$$

$$\rightarrow \left[\begin{array}{cccc:c} 1 & 1 & 1 & 1 & 6 \\ 0 & -1 & -1 & 1 & 0 \\ 0 & 2 & -1 & -4 & -13 \\ 0 & -2 & -5 & -1 & -15 \end{array}\right] \begin{array}{l} \\ ① \leftarrow \text{new pivotal row} \\ ② \quad ②+2\cdot① \rightarrow \; 0 \;\; -3 \;\; -2 \;\vdots\; -13 \\ ③ \quad ③-2① \rightarrow \; 0 \;\; -3 \;\; -3 \;\vdots\; -15 \end{array}$$

$$\rightarrow \left[\begin{array}{cccc:c} 1 & 1 & 1 & 1 & 6 \\ 0 & -1 & -1 & 1 & 0 \\ 0 & 0 & -3 & -2 & -13 \\ 0 & 0 & -3 & -3 & -15 \end{array}\right] \begin{array}{l} \\ \\ ① \leftarrow \text{new pivotal row.} \\ ② \quad ②-① \quad 0 \;\; -1 \;\vdots\; -2 \end{array}$$

$$\rightarrow \left[\begin{array}{cccc:c} 1 & 1 & 1 & 1 & 6 \\ 0 & -1 & -1 & 1 & 0 \\ 0 & 0 & -3 & -2 & -13 \\ 0 & 0 & 0 & -1 & -2 \end{array}\right]$$

The coefficient matrix is in a strict triangular form

The augmented matrix corresponds to the following linear system,

$$\begin{aligned} x_1 + x_2 + x_3 + x_4 &= 6 \quad ① \\ -x_2 - x_3 + x_4 &= 0 \quad ② \\ -3x_3 - 2x_4 &= -13 \quad ③ \\ -x_4 &= -2 \quad ④ \end{aligned}$$

which can be solved by back substitution.

$$④ \Rightarrow \quad x_4 = 2 \xrightarrow{\text{Plug into ③}} -3x_3 - 4 = -13$$

$$\Rightarrow \quad x_3 = 3$$

$$\xrightarrow{\text{Plug into ②}} -x_2 - 3 + 2 = 0$$

$$\Rightarrow \quad x_2 = -1$$

$$\xrightarrow{\text{Plug into ①}} x_1 - 1 + 3 + 2 = 6$$

$$\Rightarrow \quad x_1 = 2$$

We obtain the unique solution $(2, -1, 3, 2)$

Remarks.

① In the concept of "triangular form", "strict" means the coefficient of x_k in row k is nonzero.

② All the operations are ROW OPERATIONS.
We only operate rows in the system or in the matrix vertically.