opHW8

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Sec 6.1, 6*,8*,17*.

来自 <https://users.math.msu.edu/users/zhangshiwen/s19/homework.html>

6. Let λ be an eigenvalue of A and let $\mathbf{x}$ be an eigenvector belonging to λ. Use mathematical induction to show that, for $m \geq 1$, λ^m is an eigenvalue of A^m and $\mathbf{x}$ is an eigenvector of A^m belonging to λ^m.

If $A\bar{x} = \lambda\bar{x}$, then

$$A^2\bar{x} = A(A\bar{x}) = A(\lambda\bar{x})$$
$$= \lambda(A\bar{x})$$
$$= \lambda\cdot\lambda\cdot\bar{x}$$
$$= \lambda^2\bar{x}$$

Therefore, λ^2 is an e-value of A^2 with e-vector $\bar{x}$.

Inductively, if $A^m\bar{x} = \lambda^m\bar{x}$,

then $A^{m+1}\bar{x} = A(A^m\bar{x})$

$$= A\cdot(\lambda^m\bar{x})$$

8. An $n \times n$ matrix A is said to be *idempotent* if $A^2 = A$. Show that if λ is an eigenvalue of an idempotent matrix, then λ must be either 0 or 1.

$= A \cdot (\lambda^m \bar{x})$

$= \lambda^m \cdot (A\bar{x})$

$= \lambda^m \cdot \lambda \bar{x}$

$= \lambda^{m+1} \cdot \bar{x}$

therefore, λ^{m+1} is an evalue of A^{m+1} with e-vector $\bar{x}$.

8. An $n \times n$ matrix A is said to be *idempotent* if $A^2 = A$. Show that if λ is an eigenvalue of an idempotent matrix, then λ must be either 0 or 1.

If $A\bar{x} = \lambda \bar{x}$

then $A^2 \bar{x} = \lambda^2 \bar{x}$ by #6.

On the other hand, $A^2 = A$ implies

$$A^2 \bar{x} = A\bar{x} = \lambda \bar{x}$$

Therefore, $\lambda^2 \bar{x} = \lambda \bar{x}$ for some $\bar{x} \neq \bar{0}$.

$\Rightarrow \lambda^2 = \lambda \Rightarrow \lambda = 0$ or 1.

8. An $n \times n$ matrix A is said to be *idempotent* if $A^2 = A$. Show that if λ is an eigenvalue of an idempotent matrix, then λ must be either 0 or 1.

17. Let λ be a nonzero eigenvalue of A and let $\mathbf{x}$ be an eigenvector belonging to λ. Show that $A^m\mathbf{x}$ is also an eigenvector belonging to λ for $m = 1, 2, \ldots$.

$A\bar{x} = \lambda\bar{x}$. By #6, $A^m\bar{x} = \lambda\bar{x}$ for any m

Then $A(A^m\bar{x}) = A^{m+1}\bar{x} = \lambda^{m+1}\bar{x}$

$= \lambda(\lambda^n\bar{x})$

$= \lambda\cdot(A^m\bar{x})$

Therefore, $A^m\bar{x}$ is an evector of A corresponding to the evalue λ.

Sec 6.3, 4*(a), 19*, 31*(a).

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4. For each of the following, find a matrix B such that $B^2 = A$.

(a) $A = \begin{pmatrix} 2 & 1 \\ -2 & -1 \end{pmatrix}$ **(b)** $A = \begin{pmatrix} 9 & -5 & 3 \\ 0 & 4 & 3 \\ 0 & 0 & 1 \end{pmatrix}$

$p(\lambda) = \begin{vmatrix} 2-\lambda & 1 \\ -2 & -1-\lambda \end{vmatrix} = \lambda^2 - \lambda = 0$

$\lambda_1 = 0$, $\lambda_2 = 1$ (eigenvalues)

$\lambda_1=0$, $\lambda_2=1$ (eigenvalues)

$\lambda_1=0$.

$$A-0\cdot I=\begin{bmatrix}2 & 1\\ -2 & -1\end{bmatrix}\rightarrow\begin{bmatrix}2 & 1\\ 0 & 0\end{bmatrix}$$

e-vector of λ_1: $v_1=\begin{bmatrix}1\\ -2\end{bmatrix}$.

$\lambda_2=1$.

$$A-I=\begin{bmatrix}1 & 1\\ -2 & -2\end{bmatrix}\rightarrow\begin{bmatrix}1 & 1\\ 0 & 0\end{bmatrix}$$

e-vector of λ_2: $v_2=\begin{bmatrix}1\\ -1\end{bmatrix}$.

Let $X=(v_1,v_2)=\begin{bmatrix}1 & 1\\ -2 & -1\end{bmatrix}$, $X^{-1}=\begin{bmatrix}-1 & -1\\ 2 & 1\end{bmatrix}$

and $D=\begin{bmatrix}0 & 0\\ 0 & 1\end{bmatrix}$

Then

$$A=X\cdot D\cdot X^{-1}$$

Let $C:=\sqrt{D}=\begin{bmatrix}\sqrt{0} & 0\\ 0 & \sqrt{1}\end{bmatrix}=D$

and $B = X \cdot C \cdot X^{-1} = X \cdot D X^{-1} = A = \begin{bmatrix} 2 & 1 \\ -2 & -1 \end{bmatrix}$

Then $B^2 = X \cdot C^2 \cdot X^{-1} = X D X^{-1} = A$.

19. Show that if A and B are two $n \times n$ matrices with the same diagonalizing matrix X, then $AB = BA$.

Let $D_1 = \begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{bmatrix}$ and $D_2 = \begin{bmatrix} \mu_1 & & \\ & \ddots & \\ & & \mu_n \end{bmatrix}$

Suppose

$$A = X D_1 \cdot X^{-1} \quad \text{and} \quad B = X \cdot D_2 \cdot X^{-1}$$

Then

$$AB = X \cdot D_1 \cdot X^{-1} \cdot X \cdot D_2 \cdot X^{-1}$$

$$= X \cdot D_1 \cdot D_2 \cdot X^{-1}$$

$$BA = X \cdot D_2 \cdot X^{-1} \cdot X \cdot D_2 \cdot X^{-1}$$

$$= X \cdot D_2 \cdot D_1 \cdot X^{-1}$$

It is easy to check that.

$$D_1 D_2 = \begin{bmatrix} \lambda_1\mu_1 & & \\ & \ddots & \\ & & \lambda_n\mu_n \end{bmatrix} = D_2 \cdot D_1$$

Therefore, $AB = BA$.

31. Compute e^A for each of the following matrices:

(a) $A = \begin{bmatrix} -2 & -1 \\ 6 & 3 \end{bmatrix}$ **(b)** $A = \begin{bmatrix} 3 & 4 \\ -2 & -3 \end{bmatrix}$

$e^A = X e^D X^{-1}$ if $A = X D X^{-1}$

and $e^D = \begin{bmatrix} e^{\lambda_1} & & \\ & \ddots & \\ & & e^{\lambda_n} \end{bmatrix}$ for $D = \begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda \end{bmatrix}$

$$p(\lambda) = \begin{vmatrix} -2-\lambda & -1 \\ 6 & 3-\lambda \end{vmatrix} = \lambda^2 - \lambda = 0$$

$$\lambda_1 = 0, \quad \lambda_2 = 1$$

$$A - 0 \cdot I = \begin{bmatrix} -2 & -1 \\ 6 & 3 \end{bmatrix} \to \begin{bmatrix} 2 & 1 \\ 0 & 0 \end{bmatrix}$$

$v_1 = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$ e-vector of $\lambda_1 = 0$

$$A - I = \begin{bmatrix} -3 & -1 \\ 6 & 2 \end{bmatrix} \to \begin{bmatrix} 3 & 1 \\ 0 & 0 \end{bmatrix}$$

$$V_2 = \begin{bmatrix} 1 \\ -3 \end{bmatrix} \quad \text{e-vector of } \lambda_2 = 1.$$

let $X = (V_1, V_2) = \begin{bmatrix} 1 & 1 \\ -2 & -3 \end{bmatrix}$, $X = \frac{1}{-3+2} \cdot \begin{bmatrix} -3 & -1 \\ +2 & 1 \end{bmatrix}$

$$= \begin{bmatrix} 3 & 1 \\ -2 & -1 \end{bmatrix}.$$

$$D = \begin{bmatrix} 0 & \\ & 1 \end{bmatrix}$$

let $e^D = \begin{bmatrix} e^0 & 0 \\ 0 & e^1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & e \end{bmatrix}$

then

$$e^A = X \cdot e^D X^{-1} = \begin{bmatrix} 1 & 1 \\ -2 & -3 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 \\ 0 & e \end{bmatrix} \begin{bmatrix} 3 & 1 \\ -2 & -1 \end{bmatrix}.$$