opHW5-2

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Sec 2.2, Optional ones: 12*, 16*, 15*(Hint: use Theorem 2.2.1 to claim the existence of the inverse of A), 16*, 18,19****

来自 <https://users.math.msu.edu/users/zhangshiwen/s19/homework.html>

12. Consider the 3×3 Vandermonde matrix

$$V = \begin{bmatrix} 1 & x_1 & x_1^2 \\ 1 & x_2 & x_2^2 \\ 1 & x_3 & x_3^2 \end{bmatrix}$$

(a) Show that $\det(V) = (x_2 - x_1)(x_3 - x_1)(x_3 - x_2)$.
Hint: Make use of row operation III.

(b) What conditions must the scalars x_1, x_2, and x_3 satisfy in order for V to be nonsingular?

(a). $\begin{bmatrix} 1 & x_1 & x_1^2 \\ 1 & x_2 & x_2^2 \\ 1 & x_3 & x_3^2 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & x_1 & x_1^2 \\ 0 & x_2-x_1 & x_2^2-x_1^2 \\ 0 & x_3-x_2 & x_3^2-x_2^2 \end{bmatrix}$ Row② − Row①. Row③ − Row②

Therefore,

$$\det V = \det\begin{bmatrix} x_2-x_1 & x_2^2-x_1^2 \\ x_3-x_2 & x_3^2-x_2^2 \end{bmatrix} = (x_2-x_1)(x_3^2-x_2^2)-(x_3-x_2)(x_2^2-x_1^2)$$

$$= \underline{(x_2-x_1)(x_3-x_2)}(x_3+x_2) - \underline{(x_3-x_2)(x_2-x_1)}(x_2+x_1)$$

$$= (x_2-x_1)(x_3-x_2)\cdot\left[(x_3+x_2)-(x_2+x_1)\right]$$

$$= (x_2-x_1)\cdot(x_3-x_2)(x_3-x_1).$$

(b) V is nonsingular if and only if $\det V \neq 0$

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i.e. neither two of x_1, x_2, x_3 are the same

(x_1, x_2, x_3 are all distinct from each other)

15. Let A and B be $n \times n$ matrices. Prove that if $AB = I$, then $BA = I$. What is the significance of this result in terms of the definition of a nonsingular matrix?

If $AB = I$, then

$\det(AB) = \det I = 1$

$\Rightarrow \det A \cdot \det B = 1$

Therefore, neither $\det A$ nor $\det B$ can be zero

By theorem 2.2.2.

B is invertible since $\det B \neq 0$.

Therefore, B^{-1} exists.

$AB = I \Rightarrow (AB) \cdot B^{-1} = I \cdot B^{-1}$ (multiply B^{-1} from right)

$\Rightarrow A(B \cdot B^{-1}) = B^{-1}$

$\Rightarrow AI = B^{-1}$

$\Rightarrow B \cdot (A) = B B^{-1}$ (multiplying B from the left)

$\Rightarrow \quad B \cdot (A) = B B^{-1}$ (multiplying B from the left).

$\Rightarrow \quad B \cdot A = I$.

16. A matrix A is said to be *skew symmetric* if $A^T = -A$. For example,

$$A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$

is skew symmetric, since

$$A^T = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} = -A$$

If A is an $n \times n$ skew-symmetric matrix and n is odd, show that A must be singular.

$\det(A^T) = \det(-A)$

Left hand side: $\det(A^T) = \det A$.

Right hand side: $\det(-A) = (-1)^n \det A$

$\Rightarrow \det A = (-1)^n \cdot \det A$.

If n is odd, then $(-1)^n = -1$.

$\Rightarrow \det A = -\det A \Rightarrow \det A = 0$

$\Rightarrow A$ is singular.

18. Let A be a $k \times k$ matrix and let B be an $(n-k) \times (n-k)$ matrix. Let

$$E = \begin{bmatrix} I_k & O \\ O & B \end{bmatrix}, \quad F = \begin{bmatrix} A & O \\ O & I_{n-k} \end{bmatrix},$$

$$C = \begin{bmatrix} A & O \\ O & B \end{bmatrix}$$

where I_k and I_{n-k} are the $k \times k$ and $(n-k) \times (n-k)$ identity matrices.

(a) Show that $\det(E) = \det(B)$.

(b) Show that $\det(F) = \det(A)$.

(c) Show that $\det(C) = \det(A)\det(B)$.

(a). $\det E = \det \left[\begin{array}{ccc|c} 1 & & 0 & \\ & \ddots & & O \\ 0 & & 1 & \\ \hline & O & & B \end{array}\right]$ (Expand w.r.t. the left upper entry / inductively)

$= \underbrace{1 \cdot 1 \cdot 1 \cdots 1}_{k \text{ times}} \cdot \det B$

(b). $\det F = \det \left[\begin{array}{c|ccc} A & & O & \\ \hline & 1 & & 0 \\ O & & \ddots & \\ & 0 & & 1 \end{array}\right]$ Expand w.r.t the right bottom entry

$= \underbrace{1 \cdot 1 \cdots \cdots 1}_{n-k \text{ times}} \cdot \det A.$

(c). Direct computation shows.

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$$E \cdot F = \left[\begin{array}{c|c} I_k & 0 \\ \hline 0 & B \end{array}\right] \cdot \left[\begin{array}{c|c} A & 0 \\ \hline 0 & I_{n-k} \end{array}\right]$$

$B = (b_{ij})$ $\quad A = (a_{ij})$

$m = n-k$

$$= \left[\begin{array}{ccc|ccc} 1 & & 0 & & & \\ & \ddots & & & 0 & \\ 0 & & 1 & & & \\ \hline & & & b_{11} & \cdots & b_{1m} \\ & 0 & & \vdots & & \\ & & & b_{n1} & \cdots & b_{nn} \end{array}\right] \cdot \left[\begin{array}{ccc|ccc} a_{11} & \cdots & a_{1k} & & & \\ \vdots & & & & 0 & \\ a_{k1} & \cdots & a_{kk} & & & \\ \hline & & & 1 & & 0 \\ & 0 & & & \ddots & \\ & & & 0 & & 1 \end{array}\right]$$

$$= \begin{bmatrix} a_{11} & & a_{1k} & & & \\ & \ddots & & & 0 & \\ a_{k1} & & a_{kk} & & & \\ & & & b_{11} & \cdots & b_{1m} \\ & 0 & & \vdots & & \\ & & & b_{m1} & \cdots & b_{nm} \end{bmatrix}$$

$$= \begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix} = C.$$

Therefore, $\det C = \det E \cdot \det F$

$= \det A \cdot \det B$.