opHW5-1

2019年2月15日 10:14

Sec 2.1, Optional ones: 8*, 13*, and the case n=2 for 10*

来自 <https://users.math.msu.edu/users/zhangshiwen/s19/homework.html>

8. Write out the details of the proof of Theorem 2.1.3.

Theorem 2.1.3 *If A is an $n \times n$ triangular matrix, then the determinant of A equals the product of the diagonal elements of A.*

Proof. Suppose $A = \begin{bmatrix} a_{11} & \cdots & & a_{1n} \\ 0 & a_{22} & \cdots & a_{2n} \\ \vdots & 0 & \ddots & \\ 0 & 0 & 0 & a_{nn} \end{bmatrix}$, which is upper triangular.

Expand A w.r.t. its first column, we have

$$\det A = a_{11} \cdot \det \begin{bmatrix} a_{22} & \cdots & a_{2n} \\ 0 & a_{33} \cdots & a_{3n} \\ \vdots & 0 \ddots & \vdots \\ 0 & 0 \cdots & a_{nn} \end{bmatrix}$$

Notice that the new determinant is again in upper triangular form.

Expand w.r.t. the left-upper corner entries inductively, we have.

$$\det A = a_{11} \cdot \det \begin{bmatrix} a_{22} & \cdots & a_{2n} \\ 0 & a_{33} \cdots & a_{3n} \\ \vdots & 0 \ddots & \\ 0 & 0 \cdots & a_{nn} \end{bmatrix}$$

$$= a_{11} \cdot a_{22} \cdot \det \begin{bmatrix} a_{33} & \cdots & a_{3n} \\ 0 & a_{44} & \\ \vdots & & \ddots \\ 0 & \cdots & a_{nn} \end{bmatrix}$$

$$\cdots\cdots = a_{11} \cdot a_{22} \cdot a_{33} \cdots\cdots a_{nn}.$$

the lower triangular form can be proved in the same way by expanding w.r.t the first row.

10. Use mathematical induction to prove that if A is an $(n+1) \times (n+1)$ matrix with two identical rows, then $\det(A) = 0$.

Proof the case $n=2$. ie. A is a 3×3 matrix with two identical rows.

Suppose $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{21} & a_{22} & a_{23} \end{bmatrix}$ has the same 2nd and 3rd rows.

Then. $\det A = a_{11} \cdot \begin{vmatrix} a_{22} & a_{23} \\ a_{22} & a_{23} \end{vmatrix} - a_{12} \cdot \begin{vmatrix} a_{21} & a_{23} \\ a_{21} & a_{23} \end{vmatrix} + a_{13} \cdot \begin{vmatrix} a_{21} & a_{22} \\ a_{21} & a_{22} \end{vmatrix}$

$$= a_{11} \cdot 0 - a_{12} \cdot 0 + a_{13} \cdot 0$$

$$= 0$$

If the two identical rows are the 1st and the 2nd or the 1st and the 3rd, then we expand w.r.t. the last row in the same way as above

13. Let A be a symmetric tridiagonal matrix (i.e., A is symmetric and $a_{ij} = 0$ whenever $|i - j| > 1$). Let B be the matrix formed from A by deleting the first two rows and columns. Show that

$$\det(A) = a_{11}\det(M_{11}) - a_{12}^2\det(B)$$

$$A = \begin{bmatrix} a_{11} & a_{12} & 0 & \cdots & 0 \\ a_{12} & a_{22} & a_{23} & \cdots & 0 \\ 0 & a_{23} & \ddots & & \vdots \\ \vdots & 0 & & \ddots & \\ 0 & 0 & \cdots & & a_{nn} \end{bmatrix}, \quad B = \begin{bmatrix} a_{33} & a_{34} & \cdots & 0 \\ a_{34} & \ddots & & \\ 0 & & \ddots & \\ \vdots & & & \\ 0 & & \cdots & a_{nn} \end{bmatrix}$$

Expand w.r.t. the first row.

$$\det A = a_{11}\cdot\det M_{11} - a_{12}\cdot\det M_{12}, \text{ where}$$

$$\det M_{12} = \det \begin{bmatrix} a_{12} & a_{23} & \cdots & 0 \\ 0 & a_{33} & \ddots & \\ \vdots & & & \end{bmatrix} \Rightarrow B$$

$$\det M_{12} = \det \begin{bmatrix} 0 & \\ \vdots & \underbrace{\begin{bmatrix} a_{33} & \cdots & \\ \vdots & \ddots & \\ 0 & \cdots & a_{nn} \end{bmatrix}}_{B} \\ 0 & \end{bmatrix}$$

$$= a_{12} \cdot \det B \qquad \text{(expand with the first column).}$$

Therefore, $\det A = a_{11} \cdot \det M_{11} - a_{12} \cdot a_{12} \cdot \det B$