opHW12

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Sec 5.5, 12*, 14*, 15*

来自 <https://users.math.msu.edu/users/zhangshiwen/s19/homework.html>

12. If Q is an $n \times n$ orthogonal matrix and $\mathbf{x}$ and $\mathbf{y}$ are nonzero vectors in $\mathbb{R}^n$, then how does the angle between $Q\mathbf{x}$ and $Q\mathbf{y}$ compare with the angle between $\mathbf{x}$ and $\mathbf{y}$? Prove your answer.

The angles are the same since

$$\langle Q\bar{x}, Q\bar{y} \rangle = \langle \bar{x}, \bar{y} \rangle$$

and $\|Q\bar{x}\| = \|\bar{x}\|$, $\|Q\bar{y}\| = \|\bar{y}\|$

$$\cos\theta_1 = \frac{\langle Q\bar{x}, Q\bar{y} \rangle}{\|Q\bar{x}\| \cdot \|Q\bar{y}\|} = \frac{\langle \bar{x}, \bar{y} \rangle}{\|\bar{x}\| \cdot \|\bar{y}\|} = \cos\theta_2$$

14. Let $\mathbf{u}$ be a unit vector in $\mathbb{R}^n$ and let $H = I - 2\mathbf{u}\mathbf{u}^T$. Show that H is both orthogonal and symmetric and hence is its own inverse.

Recall that H is orthogonal if and only if

$$H^T \cdot H = I.$$

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$$H^T = (I - 2u \cdot u^T)^T$$

$$= I^T - (2 \cdot u\, u^T)^T$$

$$= I - 2 \cdot (u^T)^T \cdot u^T$$

$$= I - 2 \cdot u \cdot u^T = H \quad (H \text{ is symmetric})$$

$$H^T \cdot H = H \cdot H = (I - 2u \cdot u^T)(I - 2u \cdot u^T)$$

$$= (I - 2u \cdot u^T) \cdot I - (I - 2u \cdot u^T) 2 \cdot u\, u^T$$

$$= I - 2u \cdot u^T - 2 \cdot u \cdot u^T + 4 \cdot u \cdot u^T u \cdot u^T$$

$$= I - 4u \cdot u^T + 4u \cdot (u^T u) \cdot u^T$$

Notice that $u^T \cdot u = \|u\|^2 = 1$, ($u$ is unit vector)

Therefore,

$$H^T H = I - 4u \cdot u^T + 4 \cdot u \cdot 1 \cdot u^T = I.$$

H is orthogonal

$\cdots = I.$

H is orthogonal.

15. Let Q be an orthogonal matrix and let $d = \det(Q)$. Show that $|d| = 1$.

Q is orthogonal $\Leftrightarrow$ $Q^T \cdot Q = I$

Take determinant both sides.

$\det(Q^T \cdot Q) = \det I = 1$.

$$\det(Q^T Q) = \det(Q^T) \cdot \det(Q)$$
$$= \det(Q) \cdot \det(Q)$$
$$= d \cdot d = d^2$$

ie. $d^2 = 1 \Rightarrow |d| = 1$.