Practice Mid 2, Sec13

1[*Sec2.9, Linear Approximation*]

- Linearization of f at a: $L(x) = f(a) + f'(a)(x-a)$

Q1(F16): Use a linearization at $a = 9$ to find a good approximation of $\sqrt{8.99}$

Take $f(x) = \sqrt{x}$.

Then $L(8.99) \approx f(8.99) = \sqrt{8.99}$.

$f(x) = \sqrt{x}$, $f'(x) = (x^{\frac{1}{2}})' = \frac{1}{2} \cdot x^{-\frac{1}{2}}$.

$a = 9$. $f(a) = \sqrt{9} = 3$. $f'(9) = \frac{1}{2} \cdot 9^{-\frac{1}{2}}$

$= \frac{1}{2} \cdot \frac{1}{\sqrt{9}} = \frac{1}{2 \cdot 3} = \frac{1}{6}$

$L(x) = f(9) + f'(9) \cdot (x-9)$

$= 3 + \frac{1}{6}(x-9)$

Let $x = 8.99$,

$L(8.99) = 3 + \frac{1}{6}(8.99-9) = 3 + \frac{1}{6}(-0.01) = \boxed{3 - \frac{1}{600}}$

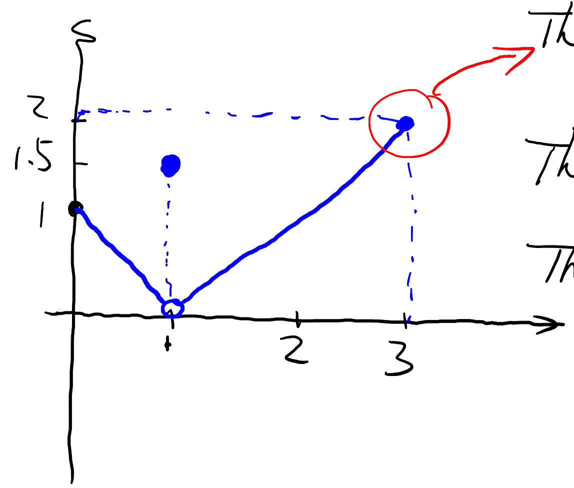
2[*Sec3.1, Extreme Values*]

- ***Extremal Value Theorem***: If $f(x)$ is continuous on the closed, finite interval $x \in [a, b]$, then $f(x)$ possesses at least one maximum point and one minimum point.
- ***Critical points***: For a function $f(x)$, a critical point (or critical number) is a point $x = c$ where the derivative is either zero or the function is not differentiable: $f'(c) = 0$ or f' undefined

Q2.1(Quiz6): Find the absolute maximum and absolute minimum values of $y = f(x)$ on the interval $[0, 3]$, where

$$f(x) = \begin{cases} |x-1|, & x \in [0,1) \cup (1,3], (i.e., x \neq 1) \\ 1.5, & x = 1 \end{cases}$$

(Hint: sketch the graph of $y = f(x)$)



The absolute maximum is $\boxed{2}$, attained at $\boxed{x=3}$

There is no absolute minimum on $[0,3]$.
This does not violate Extremal Value Theorem since the function $f(x)$ is not continuous at $x=1$.

Q2.2(F16): Find the critical numbers (i.e., critical points) of the function

$$f(x) = x^{3/2} + \frac{6}{\sqrt{x}}$$

$f(x) = x^{\frac{3}{2}} + 6 \cdot x^{-\frac{1}{2}}$

$f'(x) = \frac{3}{2} \cdot x^{\frac{3}{2}-1} + 6 \cdot (-\frac{1}{2}) \cdot x^{-\frac{1}{2}-1}$

$= \frac{3}{2} \cdot x^{\frac{1}{2}} - 3 \cdot x^{-\frac{3}{2}}$

$= \frac{3}{2} \cdot x^{\frac{1}{2}} - 3 \cdot \frac{1}{x^{\frac{3}{2}}} = 0$

$\Rightarrow \frac{3}{2} \cdot x^{\frac{1}{2}} = 3 \cdot \frac{1}{x^{\frac{3}{2}}}$ multiply by $x^{\frac{3}{2}}$

$\Rightarrow \frac{\cancel{3}}{2} x^{\frac{1}{2}} \cdot x^{\frac{3}{2}} = \cancel{3}$

$\Rightarrow \frac{1}{2} \cdot x^{\frac{1}{2}+\frac{3}{2}} = 1$

$\Rightarrow x^2 = 2$

$\Rightarrow x = \pm\sqrt{2}$

Notice the domain of f is $(0, +\infty)$

The critical point is $\boxed{x=\sqrt{2}}$

Rmk: At $x=0$, $f'(x) = \frac{3}{2} x^{\frac{1}{2}} - \frac{3}{x^{\frac{3}{2}}}$ DNE.
but $x=0$ is not in the domain, not a critical point

3[*Sec3.2, Mean Value Theorem*]

- ***(MVT)*** If f is continuous on $[a, b]$ and differentiable on (a, b) then there exists $c \in (a, b)$ that satisfies $f'(c) = \frac{f(b)-f(a)}{b-a}$

Q3(F16): If the Mean Value Theorem is applied to the function $f(x) = x^2 - 2x$ on the interval $[1, 4]$, what value of c satisfies the conclusion of the theorem in this case?

Apply $f'(c) = \frac{f(b)-f(a)}{b-a}$ to $f(x)=x^2-2x$, $[\underset{\overset{\uparrow}{a}}{1}, \underset{\overset{\uparrow}{b}}{4}]$

$f'(x) = 2x-2$. $a=1$, $b=4$.

$f(b) = f(4) = 4^2 - 2 \cdot 4 = 16 - 8 = 8$.

$f(a) = f(1) = 1^2 - 2 \cdot 1 = 1 - 2 = -1$.

$$\frac{f(b)-f(a)}{b-a} = \frac{f(4)-f(1)}{4-1} = \frac{8-(-1)}{4-1} = \frac{9}{3} = 3 = f'(c) = 2c-2.$$

$3 = 2c - 2$. solve for c.

$5 = 2c \Rightarrow \boxed{c = \frac{5}{2}}$

4[*Sec3.3, Derivatives and Graphs*]

- ***Increasing/Decreasing Theorem***: Let $f(x)$ be continuous on $[a, b]$.
  - If $f'(x) > 0$ for all $x \in (a, b)$, then $f(x)$ is increasing on $[a, b]$.
  - If $f'(x) < 0$ for all $x \in (a, b)$, then $f(x)$ is decreasing on $[a, b]$.
- ***First Derivative Test***: Let $f(x)$ be a function differentiable in a small interval around $x = c$, with $f'(c) = 0$
  - If $f'(x) < 0$ for $x < c$ and $f'(x) > 0$ for $x > c$, then $x = c$ is a local maximum of $f(x)$.
  - If $f'(x) > 0$ for $x < c$ and $f'(x) < 0$ for $x > c$, then $x = c$ is a local minimum of $f(x)$.
- ***Concavity Theorem***: Let $f(x)$ be a function.
  - If $f''(x) > 0$ for all $x \in (a, b)$, then $f(x)$ is concave up over (a, b).
  - If $f''(x) < 0$ for all $x \in (a, b)$, then $f(x)$ is concave down over (a, b).
  - If $f''(x) = 0$ and $f''(x)$ changes its sign at $x = c$, then $f(x)$ has an inflection point at $x = c$.

Q4.1(F16): Suppose

$$f(x) = \frac{x}{x^2+1}, \quad f'(x) = \frac{1-x^2}{(x^2+1)^2}, \quad f''(x) = \frac{2(x^3-3x)}{(x^2+1)^3}$$

Answer the following questions or enter none in the case of no answer.

(a) Find the largest interval(s) where f is increasing and the largest interval(s) where f is decreasing. Express your answers using interval notation.

Increasing $\Leftrightarrow f'>0$. Decreasing $\Leftrightarrow f'<0$.

$$f'(x) = \frac{1-x^2}{(x^2+1)^2} = \frac{(1-x)\cdot(1+x)}{(x^2+1)^2} = 0 \Rightarrow x = \pm 1$$

(Notice that the denominator $(x^2+1)^2$ is always positive)

f' $- - -$ -1 $+ + +$ 1 $- - -$

$f'(-2)<0$ $\quad -1 \quad$ $f'(0)>0$ $\quad 1 \quad$ $f'(2)<0$

Increasing : $[-1, 1]$

Decreasing : $(-\infty, -1] \cup [1, +\infty)$

(b) Find the interval(s) where f is concave up and the interval(s) where f is concave down. Express your answers using interval notation.

Concave up $\Leftrightarrow f''>0$, Concave down $\Leftrightarrow f''<0$

$$f''(x) = \frac{2(x^3-3x)}{(x^2+1)^3} = \frac{2x\cdot(x^2-3)}{(x^2+1)^3} = \frac{2x\cdot(x-\sqrt{3})\cdot(x+\sqrt{3})}{(x^2+1)^3}$$

Denominator $(x^2+1)^3$ is always positive. Only need to consider the numerator $2x\cdot(x-\sqrt{3})\cdot(x+\sqrt{3}) = 0 \Rightarrow x=0, x=\pm\sqrt{3}$

$f''(x)$ $- - -$ $-\sqrt{3}$ $+ + +$ 0 $- - -$ $\sqrt{3}$ $+ + +$ x

$f''(-2)<0$ $\quad -\sqrt{3} \quad$ $f''(-1)>0$ $\quad 0 \quad$ $f''(1)<0$ $\quad \sqrt{3} \quad$ $f''(2)>0$

Concave up : $(-\sqrt{3}, 0) \cup (\sqrt{3}, +\infty)$

Concave down : $(-\infty, -\sqrt{3}) \cup (0, \sqrt{3})$

5[*Sec3.4, Limits at Infinity*]

- ***Vertical asymptote***: $x = a$ is a V.A. of $f(x)$ if $f(x) \to \pm\infty$ as $x \to a$.
- ***Horzontal asymptote***: $y = L$ is a H.A. of $f(x)$ if $f(x) \to L$(finite) as $x \to \pm\infty$
- ***Limit at infinity***:
  - ***Limit for power functions of*** x:

$$p > 0, \quad \lim_{x\to\pm\infty} x^p = \pm\infty \text{ (the sign depends on } p\text{)}, \quad \lim_{x\to\pm\infty} x^{-p} = \lim_{x\to\pm\infty} \frac{1}{x^p} = 0$$

  - ***The highest term rule:*** Keep the highest term **<u>in each brackets</u>** in the numerator and denominator. Drop all the lower order terms.

Q5(S16): Find all vertical and horizontal asymptotes of

$$f(x) = \frac{3x^2 + x - 3}{x^2 + x - 6}$$

Vertical asymptote(s): $x^2+x-6=0 \Leftrightarrow (x+3)(x-2)=0$

$\Rightarrow \boxed{x=-3,\ x=2.}$

Two. V.A. $\boxed{x=-3,\ x=2.}$ (Notice neither -3 nor 2 makes the numerator 0, which implies $\lim_{x\to-3} f(x)=\infty$, $\lim_{x\to 2} f(x)=\infty$).

Horizontal asymptote(s):

$$\lim_{x\to\infty}\frac{3x^2+x-3}{x^2+x-6} \overset{\text{Highest term rule}}{=} \lim_{x\to\infty}\cancel{\frac{3\cancel{x^2}}{\cancel{x^2}}} = 3 \text{ (finite)}$$

One H.A. $\boxed{y=3}$.

6[*Sec3.5, Curve Sketching*]

- ***Slant asymptote:*** If a rational function $f(x) = mx + b + \frac{r(x)}{d(x)}$ via polynomial long(short) division and
$$\lim_{x\to\pm\infty} f(x) - (mx+b) = \lim_{x\to\pm\infty} \frac{r(x)}{d(x)} = 0,$$
then $y = mx + b$ is a S.A. of $f(x)$
- ***Method for Graphing:***
  1. Determine the domain of $f(x)$. Find the x-intercepts (solve for $f(x) = 0$); and compute the y-intercept $f(0)$ if there are any(may be none).
  2. Determine the derivatives $f'(x), f''(x)$ with Derivative Rules. Find all the increasing/decreasing and concave up/down intervals. Find all local max/min and inflection points if there are any.
  3. Find all vertical/horizontal/slant asymptotes.
  4. Draw all the above features on the graph.

Q6(S15,Quiz 8): Suppose

$$f(x) = \frac{x^2}{x-1}, \quad f'(x) = 1 - \frac{1}{(x-1)^2}, \quad f''(x) = \frac{2}{(x-1)^3}$$

(a-) Determine the domain of $f(x)$

(a) Find all the vertical asymptote and the slant asymptote of the curve $y = f(x)$. ***Explain why.***

(a-): $x-1 \neq 0 \Rightarrow x \neq 1$. Domain of f: $\boxed{(-\infty, 1) \cup (1, +\infty)}$

(a). Vertical asymptote: $\boxed{x=1}$ since $\lim_{x\to 1^+} \frac{x^2}{x-1} = +\infty$ $(x^2 \to 1, x-1 \to 0)$

$$\begin{array}{r|l} & x+1 \\ \hline x-1 & x^2+0+0 \\ & x^2-x \\ \hline & x+0 \\ & x-1 \\ \hline & 1 \end{array}$$

$\Rightarrow f(x) = x+1+\frac{1}{x-1}.$

Slant asymptote is $\boxed{y=x+1}$ since

$f(x)-(x+1) = \frac{1}{x-1}$ and $\lim_{x\to+\infty} \frac{1}{x-1} = 0$, i.e.,

$f(x)$ is approaching $x+1$ as x is approaching $+\infty$.

$$f(x) = \frac{x^2}{x-1}, \quad f'(x) = 1 - \frac{1}{(x-1)^2}, \quad f''(x) = \frac{2}{(x-1)^3}$$

(b) Identify the intervals over which f(x) is increasing and decreasing, concave up and concave down.

(c) Identify all points (x, y) where $f(x)$ attains its local maximum or minimum.

(d) Identify all values of x that are inflection points.

(e) By using the information in parts (a)-(d), sketch the curve of $y = f(x)$.

(b). $f'(x) = 1 - \frac{1}{(x-1)^2} = \frac{(x-1)^2 - 1}{(x-1)^2} = \frac{x^2-2x+\cancel{1-1}}{(x-1)^2} = \frac{x\cdot(x-2)}{(x-1)^2} = 0$

$\Rightarrow$ $x=0$, $x=2$. (and pay attention $x=1$ is not in the domain)

$f'(x)$ +++ --- --- +++ (0, 1, 2)

denominator $(x-1)^2$ is alway positive, only need to consider the sign of the numerator.

$\boxed{\text{Increasing: } (-\infty, 0] \cup [2, +\infty)}$ $\boxed{\text{Decreasing: } [0, 1) \cup (1, 2]}$

$f''(x) = \frac{2}{(x-1)^3}$ Notice that the sign of $f''(x)$ is determined by the sign of the denominator $(x-1)^3$. Actually, if $x<1$, then $(x-1)^3 < 0 \Rightarrow f''(x) < 0$. If $x>1$, then $(x-1)^3 > 0 \Rightarrow f''(x) > 0$.

$\boxed{\text{Concave up: } (1, +\infty)}$ $\boxed{\text{Concave down: } (-\infty, 1)}$

(c). f' changes from + to − at $x=0$

$\Rightarrow$ $\boxed{f \text{ attains local max at } x=0}$

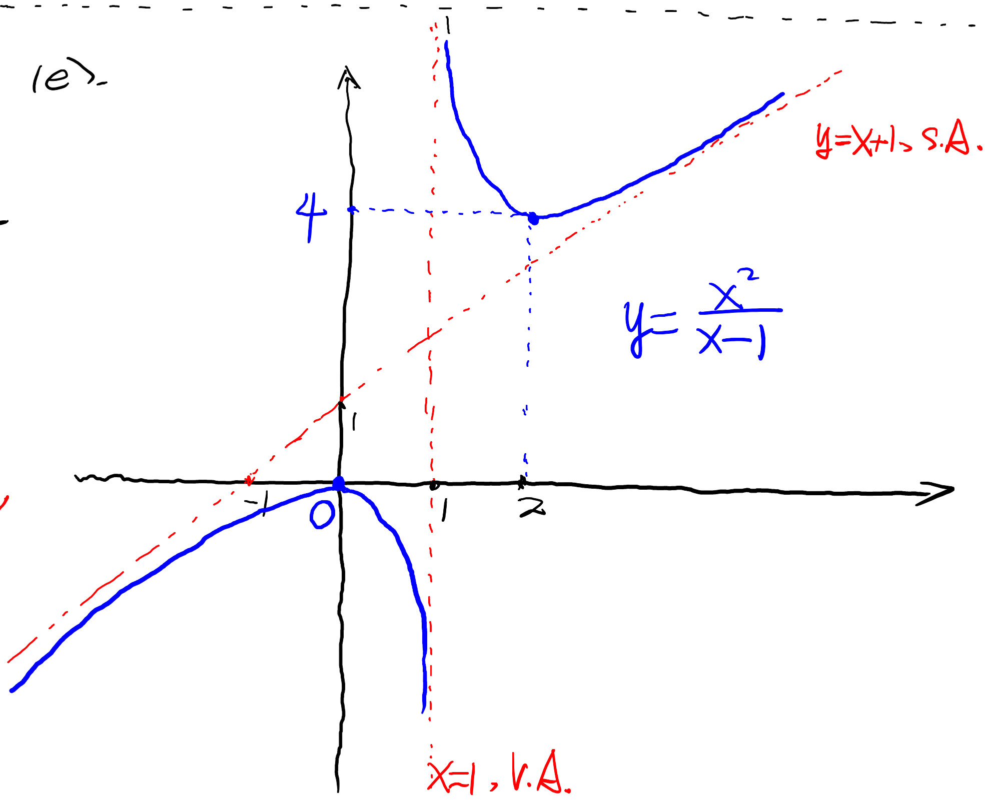
f' changes from − to + at $x=2$

$\Rightarrow$ $\boxed{f \text{ attains local min at } x=2}$

(d). There is No inflection point.

Rmk: Although $f''(x)$ changes sign at $x=1$, $x=1$ is Not in the domain of $f(x)$.

(e).



7[*Sec3.7, Optimization*]

1. Draw a picture labeled with all varying quantities. Find the target function which is to be maximized or minimized. Express the target function by other quantities.
2. Write equations relating variables. Choose one as the controlling variable, and solve for all other variables in terms of it. Plug into the target function and rewrite it using only one variable. ***Determine the domain.***
3. Find the absolute maximum/minimum of the target function.

☆ Q7(F15): A total of 1200 cm^2 of material is to be used to make a box with no top. Assume that the base is to be twice as long as it is wide.

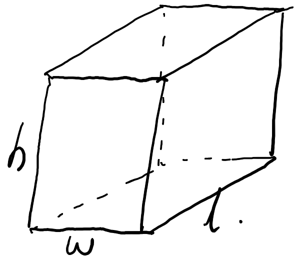
(a) Assume the dimensions of the box are as follows: height h, base length l, base width w. Express the volume of the box in terms of h, l, w.

(b) Find all relations between h, l and w. Express h, l in terms of w.

(c) Rewrite the volume V in (a) as a function of w and ***find its domain***.

(d) Find the largest volume of such a box. Explain why it is the maximum volume in the domain by determining the sign of $V'(w)$.

(a).



$$\boxed{V=h\cdot w\cdot l}$$

(b). $\boxed{l=2w}$ (base is twice as long as it is wide)

$\boxed{2h\cdot w+2h\cdot l+w\cdot l=1200}$ (total of 1200 material, no top)

Plug $l=2w$ into the second equation and then solve for h.

$2h\cdot w+2h\cdot 2w+w\cdot 2w=1200 \Leftrightarrow 6h\cdot w=1200-2w^2$

(Fix w and solve for h) $\Rightarrow \boxed{h=\frac{1200-2w^2}{6w}=\frac{600-w^2}{3w}}$

(c). $V=\frac{600-w^2}{3\cancel{w}}\cdot\cancel{w}\cdot 2w=\frac{(600-w^2)2w}{3}=\boxed{400w-\frac{2}{3}w^3}$

w, h, l have to be positive $w>0$, $h=\frac{600-w^2}{3w}>0\Rightarrow 600-w^2>0\Rightarrow 600>w^2$

$\Rightarrow \sqrt{600}>w$.

ie. w has to satisfy $\boxed{0<w<\sqrt{600}}$

(d) $V'=400-2w^2=0$

$\Rightarrow 400=2w^2$

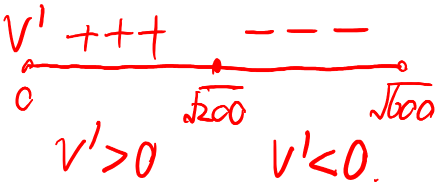
$\Rightarrow 200=w^2$

$\Rightarrow \boxed{w=\sqrt{200}}$

Notice that $V'=2(200-w^2)$

V is increasing for $w<\sqrt{200}$
and is decreasing for $w>\sqrt{200}$

Therefore, V attains max at $w=\sqrt{200}$.



The maximal volume $\boxed{V_{max}=V(\sqrt{200})}=400\cdot\sqrt{200}-\frac{2}{3}(\sqrt{200})^3=400\sqrt{200}-\frac{400}{3}\sqrt{200}=\boxed{\frac{800}{3}\cdot\sqrt{200}}$

8[*Sec3.8, Newton's Method*]

- Newton's Method: $x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$; $x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}, x_3 = x_2 - \frac{f(x_2)}{f'(x_2)}$
- The choice of $f(x)$ such that Newton's Method gives an estimate of the solution to $f(x) = 0$

Q8(F14,WW): Newton's method can be used to approximate $\sqrt[3]{7}$. Give the function $f(x)$ that Newton's Method approximates $\sqrt[3]{7}$ by finding the root of $f(x)$. Find x_2 using Newton's method with $x_1 = 2$.

$$x = \sqrt[3]{7} \iff x^3 = 7 \iff \boxed{x^3 - 7} = 0$$

i.e. $\sqrt[3]{7}$ is a solution to $x^3 - 7 = 0$,

Let $\boxed{f(x) = x^3 - 7.}$

$$f'(x) = 3 \cdot x^2$$

$x_1 = 2$ $\quad f(x_1) = f(2) = 2^3 - 7 = 8 - 7 = 1.$

$$f'(x_1) = f'(2) = 3 \cdot 2^2 = 12 .$$

$(n=1)$

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$$

$$= 2 - \frac{1}{12} = \boxed{\frac{23}{12}}$$

Q9[*Sec3.9, Antiderivatives*]

- ***Antiderivative***. $F(x)$ is an antiderivative of $f(x)$ if $F'(x) = f(x)$. $F(x) + C$ for any constant C is called the most general antiderivative of $f(x)$
- $x^n = nx^{n-1}$, $(\sin x)' = \cos x, (\cos x)' = -\sin x, (\tan x)' = \sec^2 x, (\sec x)' = \sec x \cdot \tan x$
- ***Antiderivative Table:***

$f(x)$	$x^n, n \neq -1$	$\cos x$	$\sin x$	$\sec^2 x$	$\sec x \cdot \tan x$
Anti-D F(x)	$\frac{1}{n+1}x^{n+1}$	$\sin x$	$-\cos x$	$\tan x$	$\sec x$

- $f(x)$ is the (most general) anti-D of $f'(x)$. $f(a) = b$ can be used to determine the constant C.
- Position $s(t)$ is the anti-D of velocity $v(t)$. $v(t)$ is the anti-D of acceleration $a(t)$.

Q9.1(S16): Find the general anti-derivative of

$$g(t) = \frac{3t+1}{\sqrt{t}}$$

$g(t) = \frac{3t}{t^{\frac{1}{2}}} + \frac{1}{t^{\frac{1}{2}}} = 3t^{\frac{1}{2}} + t^{-\frac{1}{2}}$

anti-D of $t^{\frac{1}{2}}$ $(n=\frac{1}{2}) = \frac{1}{\frac{1}{2}+1} \cdot t^{\frac{1}{2}+1} = \frac{2}{3}t^{\frac{3}{2}}$

anti-D of $t^{-\frac{1}{2}}$ $(n=-\frac{1}{2}) = \frac{1}{-\frac{1}{2}+1} \cdot t^{-\frac{1}{2}+1} = 2 \cdot t^{\frac{1}{2}}$

The most general anti-D of $g(t)$ is:

$$\boxed{3 \cdot \frac{2}{3} \cdot t^{\frac{3}{2}} + 2 \cdot t^{\frac{1}{2}} + C}$$

Q9.2(S16): Solve the following initial value problem: Suppose $f'(x) = -\cos x$ and $f(\pi/2) = 0$. Find $f(x)$.

$f(x)$ is the (most general) anti-D of $f'(x) = -\cos x$.

i.e. $f(x) = -\sin x + C$

Plug in $x = \frac{\pi}{2}$, $0 = f(\frac{\pi}{2}) = -\sin\frac{\pi}{2} + C$

$\Rightarrow 0 = -1 + C \Rightarrow C = 1$

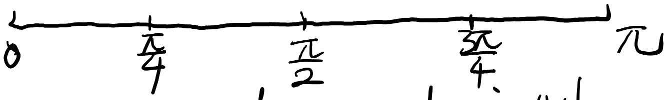
Therefore, $\boxed{f(x) = -\sin x + 1}$

10[*Sec4.1, Area and Distance*]

- Approximating the area under the curve by finite rectangles; Upper and Lower sum.
- Area/Integral under $y = f(x)$ on $[a, b]$ as the limit of a Riemann sum.

$$\text{Area} = \lim_{n\to\infty}\sum_{i=1}^{n} f(x_i)\Delta x,\ \ \Delta x = \frac{b-a}{n}, x_i = a + i\Delta x, i = 1, 2, \cdots, n$$

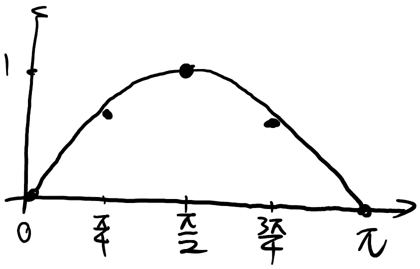
Q10(S15): Find (a) the upper sum and (b) the lower sum, when we estimate the area under the graph of $f(x) = \sin x$ from $x = 0$ to $x = \pi$ using four rectangles of equal width.



width $\Delta x = \frac{\pi - 0}{4} = \frac{\pi}{4}$.

Interval: 1st 2nd 3rd 4th

x	0	$\frac{\pi}{4}$	$\frac{\pi}{2}$	$\frac{3\pi}{4}$	π
$\sin x$	$\sin 0 = 0$	$\sin\frac{\pi}{4} = \frac{\sqrt{2}}{2}$	$\sin\frac{\pi}{2} = 1$	$\sin\frac{3\pi}{4} = \frac{\sqrt{2}}{2}$	$\sin\pi = 0$

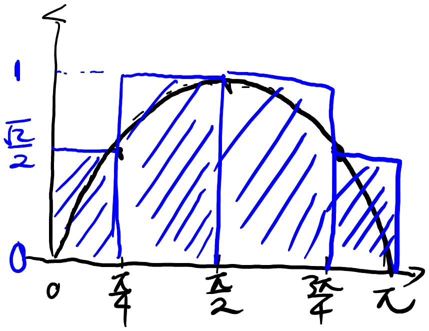


(a) Upper Sum

$= \frac{\pi}{4}\cdot f(\frac{\pi}{4}) + \frac{\pi}{4}\cdot f(\frac{\pi}{2}) + \frac{\pi}{4}\cdot f(\frac{\pi}{2}) + \frac{\pi}{4}\cdot f(\frac{3\pi}{4})$.

$= \frac{\pi}{4}\cdot\frac{\sqrt{2}}{2} + \frac{\pi}{4}\cdot 1 + \frac{\pi}{4}\cdot 1 + \frac{\pi}{4}\cdot\frac{\sqrt{2}}{2}$

$= \frac{\pi}{4}(\sqrt{2} + 2)$

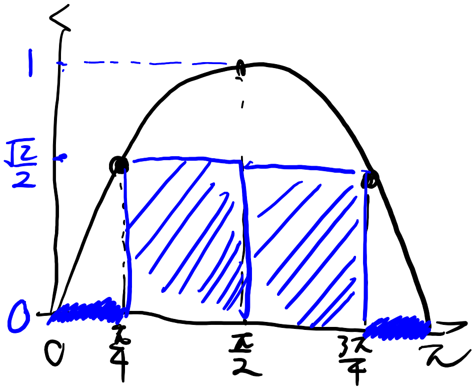


(b) Lower Sum

$= \frac{\pi}{4}\cdot f(0) + \frac{\pi}{4}\cdot f(\frac{\pi}{4}) + \frac{\pi}{4}\cdot f(\frac{3\pi}{4}) + \frac{\pi}{4}\cdot f(\pi)$

$= \frac{\pi}{4}\cdot 0 + \frac{\pi}{4}\cdot\frac{\sqrt{2}}{2} + \frac{\pi}{4}\cdot\frac{\sqrt{2}}{2} + \frac{\pi}{4}\cdot 0$

$= \frac{\pi}{4}\cdot\sqrt{2}$



11[*Sec4.2, The Definite Integral*]

- ***(Definite) Integral*** as ***Area under the curve*** and as ***the limit of a Riemann sum***

$$\int_a^b f(x)dx = \text{Area under } f(x)\text{(up to sign)} = \lim_{n\to\infty}\sum_{i=1}^n f\left(a+i\frac{b-a}{n}\right)\cdot\frac{b-a}{n}$$

- Integral Rules.

Sum/Diff/Const.Multi.: $\int_a^b f(x)\pm g(x)dx = \int_a^b f(x)dx \pm \int_a^b g(x)dx$; $\int_a^b C\cdot f(x)dx = C\cdot\int_a^b f(x)dx$

Splitting/Fliping:

$$\int_a^c f(x)dx = \int_a^{\boxed{b}} f(x)dx + \int_{\boxed{b}}^c f(x)dx;\quad \int_a^a f(x)dx = 0;\quad \int_a^b f(x)dx = -\int_b^a f(x)dx$$

- Basic integrals from the graph:

Rectangle: $\int_a^b 1dx = b-a$; $\int_a^b Cdx = C(b-a)$

Half/Quater disk: $\int_{-1}^1 \sqrt{1-x^2}dx = \frac{1}{2}\pi$; $\int_{-r}^r \sqrt{r^2-x^2}dx = \frac{1}{2}\pi r^2$; $\int_0^r \sqrt{r^2-x^2}dx = \frac{1}{4}\pi r^2$

Triangle/Trapezoid: $\int_0^b xdx = \frac{1}{2}b^2$; $\int_a^b xdx = \frac{1}{2}b^2 - \frac{1}{2}a^2$

Q11.1(F16): Which of the following definite integrals is equivalent to the following limit of a Riemann sum?

$$\lim_{n\to\infty}\sum_{i=1}^n \sqrt{8+\frac{5i}{n}}\,\frac{5}{n} \longrightarrow \frac{b-a}{n} \Rightarrow b-a=5.$$

A. $\int_8^{13}\sqrt{8+x}\;dx$; [✓ B. $\int_8^{13}\sqrt{x}\;dx$;] C. $\int_0^1\sqrt{8+5x}\;dx$; D. $\int_0^5 5\sqrt{8+x}\;dx$; E. $\int_8^{13}\sqrt[3]{8+5x}\;dx$;

$f(x_i)$, $f(\cdot)=\sqrt{\square}$, $x_i = 8+\frac{5i}{n}$

$\uparrow a$

$a=8$, $b\overset{\Leftarrow}{=}13$.

C. does not have the correct interval length.

D, E do not have the correct function $\sqrt{\cdots\cdots}$

D: $5\sqrt{\cdots}$. E: $\sqrt[3]{\cdots\cdots}$.

The only possible answers are A or B.

A: $\int_8^{13}\sqrt{8+x}\cdot dx$, $\Delta x = \frac{13-8}{n} = \frac{5}{n}$. $f(x)=\sqrt{8+x}$

$a=8$, $x_i = a+i\cdot\Delta x = 8+i\frac{5}{n}$

$\Rightarrow f(x_i) = \sqrt{8+x_i} = \sqrt{16+i\frac{5}{n}}$ does not match.

Q11.2(F16): Suppose $\int_2^5 f(x)\,dx = 3$ and $\int_2^3 f(x)\,dx = -4$. Find $\int_3^5 2f(x)\,dx$.

$\int_3^5 2f(x)\cdot dx = \int_3^2 2f(x)dx + \int_2^5 2f(x)dx$

$= -\int_2^3 2f(x)dx + \int_2^5 2f(x)dx$

$= -2\cdot\int_2^3 f(x)dx + 2\cdot\int_2^5 f(x)dx$

$= -2\cdot(-4) + 2\cdot 3 = 8+6 = \boxed{14}$

(F15): Suppose $\int_1^4 f(x)\,dx = 5$ and $\int_2^4 f(x)\,dx = 3$. Find $\int_1^2 \left(2f(x) - 3\right)\,dx$.

$\int_1^2 (2f(x)-3)dx = \int_1^2 2f(x)dx - \int_1^2 3\cdot dx$

$= \int_1^4 2f(x)dx + \int_4^2 2f(x)dx \quad - 3\cdot(2-1)$

$= 2\int_1^4 f(x)dx - 2\int_2^4 f(x)dx \quad - 3\cdot 1$

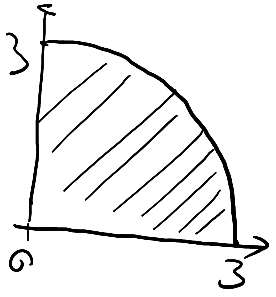
$= 2\cdot 5 \quad - 2\cdot 3 \quad - 3$

$= 10 - 6 - 3 = \boxed{1}$

Q11.3(F16): Evaluate (Hint: a definite integral represents an area.)

$$\int_0^3 \sqrt{9-x^2}dx, \quad \text{and} \quad \int_0^4 |3-x|dx$$

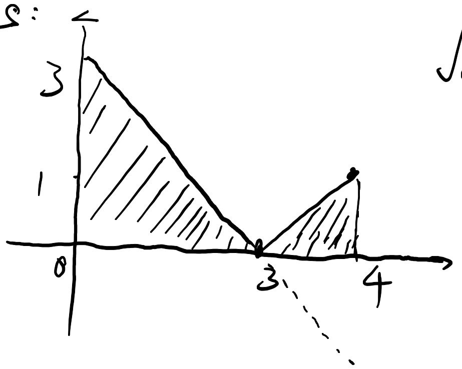
Quater Disk:



$\int_0^3 \sqrt{9-x^2}\,dx = \text{Area} = \frac{1}{4}\pi\cdot 3^2$

$= \boxed{\frac{9}{4}\pi}$

Triangles:



$\int_0^4 |3-x|\,dx = \frac{1}{2}3\cdot 3 + \frac{1}{2}\cdot 1\cdot 1$

$= \frac{9}{2} + \frac{1}{2}$

$= \boxed{5}$

12[*Sec4.3, Fundamental Theorem of Calculus*]

- ***FToC P1:*** If $F(x) = \int_a^x f(t)\, dt$, then $F'(x) = \left(\int_a^x f(t)\, dt\right)' = f(x)$.
- ***FToC P1 Chain rule form:*** $\left(\int_{v(x)}^{u(x)} f(t)\, dt\right)' = f\big(u(x)\big)\cdot u'(x) - f\big(v(x)\big)\cdot v'(x)$

$$\left(\int_a^{u(x)} f(t)\, dt\right)' = f\big(u(x)\big)\cdot u'(x), \quad \left(\int_{v(x)}^{b} f(t)\, dt\right)' = -f\big(v(x)\big)\cdot v'(x)$$

- ***FToC P2:*** If $F(x)$ is an anti-D of $f(x)$, i.e., $F'(x) = f(x)$, then $\int_a^b f(x)\, dx = F(x)|_a^b = F(b) - F(a)$
- ***Antiderivative Table:***

$f(x)$	$x^n, n \neq -1$	$\cos x$	$\sin x$	$\sec^2 x$	$\sec x \cdot \tan x$
Anti-D F(x)	$\frac{1}{n+1}x^{n+1}$	$\sin x$	$-\cos x$	$\tan x$	$\sec x$

Q12.1(F16): Let

$$F(x) = \int_{x^3}^{1} \frac{1}{t^2+2}\, dt,$$

find $F'(x)$. $f(t)=\frac{1}{t^2+2}$, $V(x)=x^3$

$$F'(x) = -f(V(x))\cdot V'(x) = -\frac{1}{(x^3)^2+2}\cdot (x^3)'$$

↑ replace t by x^3

$$= \boxed{-\frac{1}{x^6+2}\cdot 3x^2}$$

Q12.2(F15): Let

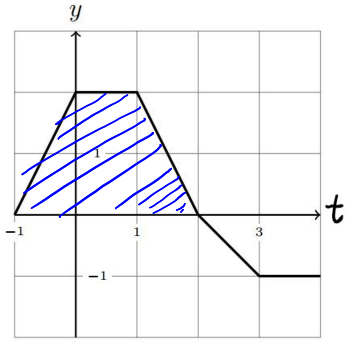
$$f(x) = \int_0^{x^2} \sqrt{1+t^2}\, dt,$$

find $f'(x)$. $\sqrt{1+t^2}$, $u(x)=x^2$

$$f'(x) = \sqrt{1+(x^2)^2}\cdot (x^2)'$$

$$= \boxed{\sqrt{1+x^4}\cdot 2x.}$$

Q12.3(S16): The graph of a function f for $-1 \leq t \leq 4$ is shown below. What is the value of $\int_{-1}^{2} f(t)\, dt$.



$\int_{-1}^{2} f(t)dt$ = Area from $t=-1$ to $t=2$ (Trapezoid)

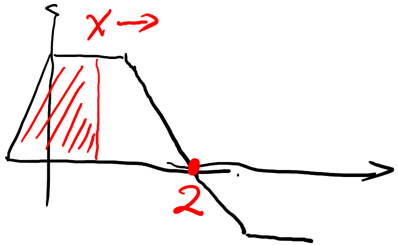
$= \frac{1}{2}\cdot 2\cdot(1+3)$

$= \boxed{4}$

Suppose $g(x) = \int_{-1}^{x} f(t)\, dt$. Find $g(-1), g(2)$. When does $g(x)$ attain its maximum on $[-1, 4]$?

$g(-1) = \int_{-1}^{-1} f(t) dt = 0$

$g(2) = \int_{-1}^{2} f(t) dt = 4.$



As x moves from left to right the area g(x) changes.

The cancellation starts from x=2 (after 2 the curve is below x-axis)

g(x) attains its maximum at $\boxed{x=2}$

Q12.4(F16): Evaluate

$$\int_{1}^{2} \frac{5 - 7t^6}{t^4}\, dt$$

Find anti-D of $f(t) = \frac{5-7t^6}{t^4}$ first.

$f(t) = \frac{5}{t^4} - \frac{7t^6}{t^4} = 5\cdot t^{-4} - 7\cdot t^2$

Anti-D: $F(t) = 5\cdot\frac{1}{-4+1}\cdot t^{-4+1} - 7\cdot\frac{1}{2+1}t^{2+1}$

$= 5\cdot\frac{1}{-3}\cdot t^{-3} - 7\cdot\frac{1}{3}t^3 = -\frac{5}{3}t^{-3} - \frac{7}{3}\cdot t^3$

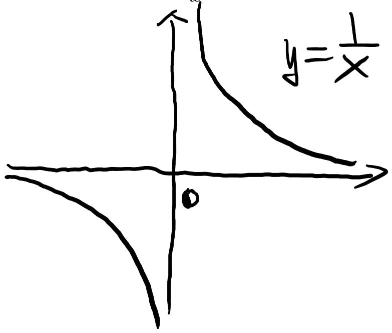
$\int_{1}^{2} \frac{5-7t^6}{t^4} dt = F(t)\Big|_1^2 = F(2) - F(1)$

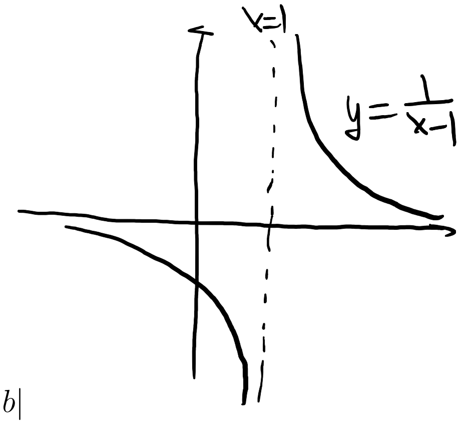
$= \boxed{(-\frac{5}{3}\cdot 2^{-3} - \frac{7}{3}\cdot 2^3) - (-\frac{5}{3}\cdot 1^{-3} - \frac{7}{3}\cdot 1^3)}$

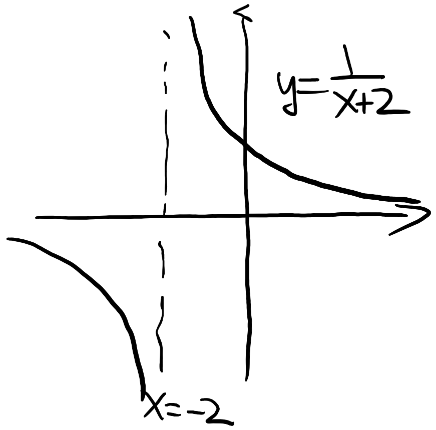
$= -\frac{5}{3}\cdot\frac{1}{8} - \frac{7}{3}\cdot 8 + \frac{5}{3} + \frac{7}{3} = -\frac{5}{24} - \frac{44}{3}$

- ***Important pre-calculus facts:***
  - $$\frac{1}{x^p} = x^{-p}, \qquad x^a \cdot x^b = x^{a+b}, \qquad \frac{x^a}{x^b} = x^{a-b} = \frac{1}{x^{b-a}}$$
  - m, n are positive integers and n is even.

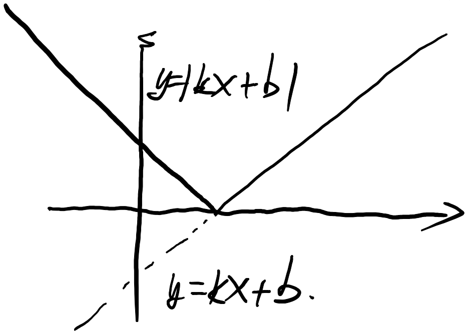
    $x^{m/n} = (\sqrt[n]{x})^m, x \geq 0$(the domain is $[0, \infty)$); $x^{-m/n} = \dfrac{1}{x^{m/n}}, x > 0$(the denominator cannot be zero)
  - Graph of $y = \frac{1}{x}$

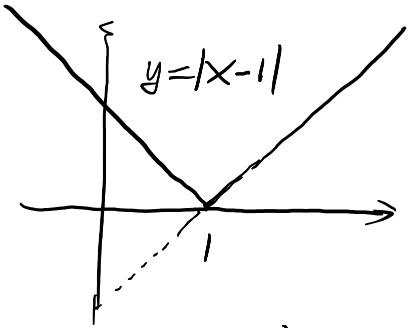


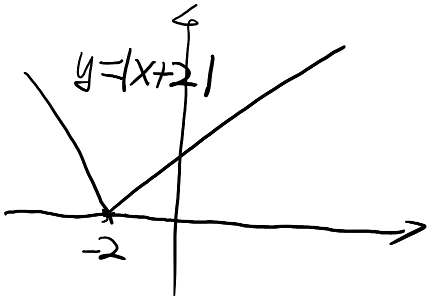




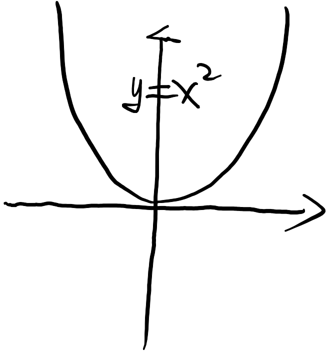
  - Graph of $y = kx + b, \quad y = |kx + b|$

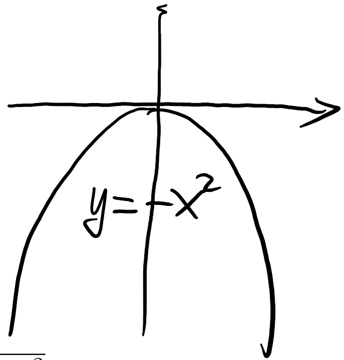


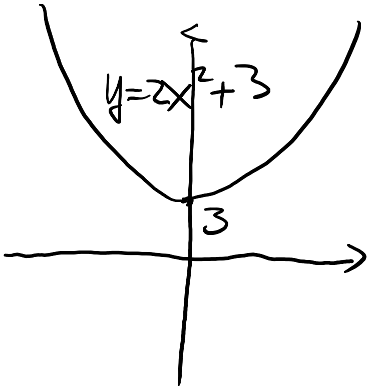


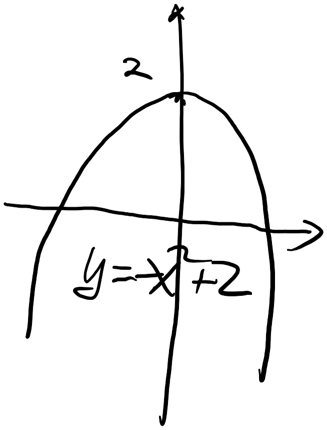


  - Graph of $y = x^2, y = -x^2$ and $y = ax^2 + c$









  - Graph of $y = \sqrt{1 - x^2}$