Practice Mid 1, Sec13

Q1[*Sec1.4, Average rate of change/Average velocity, see also Q9*] Let $f(x) = \cos x + 2$. Compute the average rate of change of $f(x)$ on the interval $[0, \frac{\pi}{2}]$

Solution: Average rate of change:

$$\begin{aligned} A.R.o.C. &= \frac{f(\pi/2) - f(0)}{\pi/2 - 0} \\ &= \frac{(\cos(\pi/2) + 2) - (\cos 0 + 2)}{\pi/2 - 0} \\ &= \frac{(0+2) - (1+2)}{\pi/2} = \frac{-1}{\pi/2} = -\frac{2}{\pi} \end{aligned}$$

□

Q2[*Sec1.5/1.6, Limit and Limit Laws*] Evaluate the following limits

(a)Direct plug in-type

Suppose $\lim_{x\to 4} f(x) = 2, \lim_{x\to 4} g(x) = 3.$ Find $\lim_{x\to 4} \frac{xf(x)+2}{f(x) - \sqrt{g(x)}}$

Solution:

$$\begin{aligned} \lim_{x\to 4} \frac{xf(x)+2}{f(x) - \sqrt{g(x)}} &= \frac{\lim_{x\to 4} x \cdot \lim_{x\to 4} f(x) + 2}{\lim_{x\to 4} f(x) - \sqrt{\lim_{x\to 4} g(x)}} \\ &= \frac{4 \cdot 2 + 2}{2 - \sqrt{3}} \\ &= \frac{10}{2 - \sqrt{3}} \end{aligned}$$

□

(b)$\frac{1}{0}$-type/One-sided limits

$$\lim_{x\to 0^+} \frac{x-3}{x^2(x+5)}$$

Solution:

$x \to 0^+ \implies x - 3 \to 0 - 3 = -3 < 0$(negative), $x^2 > 0$(positive), $x + 5 \to 0 + 5 = 5 > 0$(positive).

$$\frac{x-3}{x^2(x+5)} \sim \frac{negative}{positive \times postive} \sim negative, \implies \frac{x-3}{x^2(x+5)} \to \frac{0-3}{0^2 \cdot (0+5)} = \frac{-3}{0 \cdot 5} \to -\infty$$

$$\lim_{x\to 0^+} \frac{x-3}{x^2(x+5)} = -\infty$$

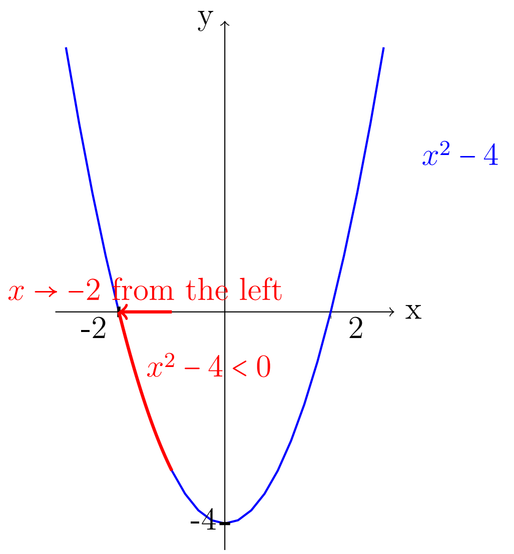
□

(c)Cancellation-type

$$\lim_{x\to -2^+} \frac{|x^2-4|}{x+2}$$

Solution: $x \to -2 \implies \frac{|x^2-4|}{x+2} \to \frac{|(-2)^2-4|}{-2+2} = \frac{|0|}{0}$, which is $\frac{0}{0}$-type. We need to cancel out the 'zero terms' then plug in $x = -2$. Before that, we need to remove the abstract value $|\cdot|$ first.

As $x \to -2^+$, $x > -2 \implies x^2 - 4 < 0 \implies |x^2 - 4| = -\left(x^2 - 4\right) = 4 - x^2$. Actually, $x^2 - 4 < 0$ follows from the graph of $y = x^2 - 4$:

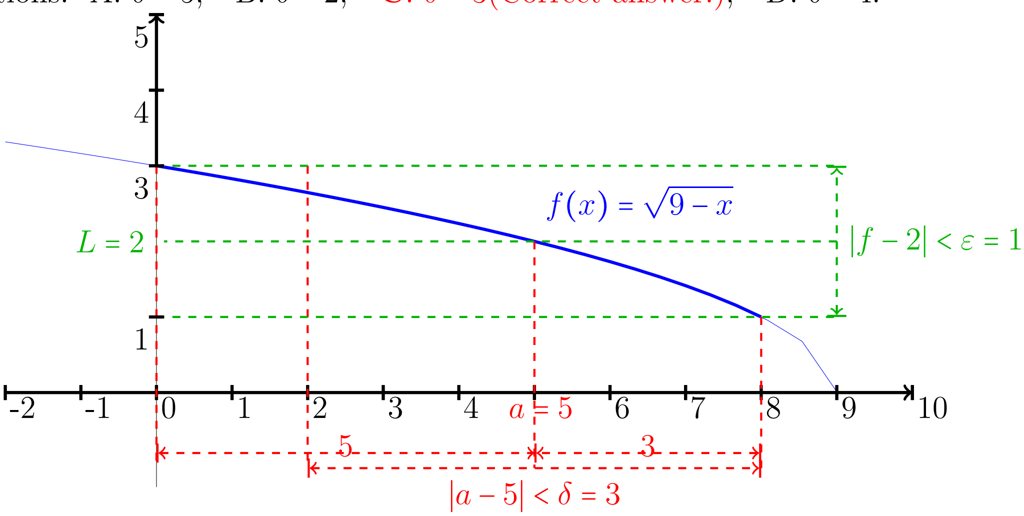


Notice that $|\blacksquare| = -\blacksquare$ if $\blacksquare < 0$. Therefore, $x^2 - 4 < 0 \Longrightarrow |x^2 - 4| = -\left(x^2 - 4\right) = 4 - x^2$. Now we have

$$\begin{aligned}
\lim_{x\to -2^+} \frac{|x^2-4|}{x+2} &= \lim_{x\to -2^+} \frac{4-x^2}{x+2}, \quad \textit{Facterization}: a^2 - b^2 = (a+b)(a-b) \\
&= \lim_{x\to -2^+} \frac{(2+x)(2-x)}{x+2}, \quad \textit{Cancel out the zero term } x+2 \\
&= \lim_{x\to -2^+} 2 - x = 2 - (-2) = 4. \quad \textit{Plug in } x = -2
\end{aligned}$$

□

Q3[*Sec1.7, Limit Definition*] For $f(x) = \sqrt{9-x}; L = 2, a = 5, \varepsilon = 1$, use the graph of $f(x)$ to find the largest value of δ of $|x - a| < \delta$ in the formal definition of a limit which ensures that $|f(x) - L| < \varepsilon$.
Options: A. $\delta = 5$; B. $\delta = 2$; **C.** $\delta = 3$(Correct answer.); D. $\delta = 4$.



Solution: $|f(x) - L| < \varepsilon = 1$ gives us the vertical green window for y from 1 to 3 (vertically). Then the intersection of the green window with the blue curve gives us the horizontal window for x from 0 to 8. The distance to $a = 5$ is 5 on the left hand side and 3 on the right hand side. We need to pick an interval for x centered at $a = 5$ and with maximum radius δ in this red window. Therefore, the maximum δ would be 3. (If you choose $\delta = 5$, the interval $|x - 5| < 5$ will exceed the red window on the right hand side. Then the corresponding $f(x)$ will escape the vertical green window.) □

Q4[*Sec1.8, Domain of continuity*] Use interval notation to indicate where $f(x)$ is continuous.

(a)

$$f(x) = \frac{x^2 - 3x + 1}{x - 3}.\quad \text{Choose from below}$$

A. $(-\infty, +\infty)$; **B**. $(-\infty, 3) \cup (3, +\infty)$; **C**. $(-\infty, 1) \cup (1, +\infty)$; **D**. $(-\infty, 1) \cup (1, 3) \cup (3, +\infty)$.

Solution: $f(x)$ is continuous everywhere in its domain. The domain of $f(x)$ is all those x such that $f(x)$ is computable(meaningful/finite number). The only point not in f's domain is $x = 3$, which makes the denominator zero. Therefore, $f(x)$ is continuous everywhere except $x = 3$. □

(b)

$$f(x) = \sqrt{x+1}.\quad \text{Choose from below}$$

A. $(-\infty, +\infty)$; **B**. $(-\infty, -1]$; **C**. $[-1, +\infty)$; **D**. $(1, +\infty)$.

Solution: Similar to part (a), $f(x)$ is continuous everywhere in its domain. The expression under square root has to be nonnegative, i.e., $x + 1 \geq 0 \Longrightarrow x \geq -1 \Longrightarrow x \in [-1, +\infty)$. □

(c)

$$f(x) = \frac{(x^2 - 3x + 1)\sqrt{x+1}}{x - 3}.\quad \text{Use (a,b) to indicate the intervals of continuous for (c)}$$

Solution: The function contains both expression in (a) and (b). Therefore, the domain where $f(x)$ where it is continuous should satisfy both (a) and (b). Combine part (a) and part (b), we have the answer $[-1, 3) \cup (3, +\infty)$. □

Q5[*Sec1.8, Piecewise function*] For what value of k will $f(x)$ be continuous for all values of x ?

$$f(x) = \begin{cases} \frac{x^2-3k}{x-3}, & x \leq 2 \\ 8x - k, & x > 2 \end{cases}$$

Options: **A**. $k = 2$; **B**. $k = 3$; **C**. $k = 4$; **D**. $k = 5$.

Solution: $f(x)$ is a piecewise function which might have a break at the connecting point $x = 2$. The strategy is simply to plug $x = 2$ into the first and second expression of f. Then set them equal and solve for k.

Plug $x = 2$ into $\frac{x^2-3k}{x-3}$, we get $\frac{2^2-3k}{2-3} = \frac{4-3k}{-1} = -(4 - 3k) = 3k - 4$.
Plug $x = 2$ into $8x - k$, we get $8x - k = 8 \cdot 2 - k = 16 - k$.
Set them equal: $3k - 4 = 16 - k \Longrightarrow 4k = 20 \Longrightarrow k = 5$.

The reason why these three steps give us the k such that f is continuous is as follows: $f(x)$ is continuous at $x = 2$ if and only if

$$(*) \quad f(2) = \lim_{x \to 2^-} f(x) = \lim_{x \to 2^+} f(x)$$

It graphically means that the left part of the curve and the right part of the curve are connected at $x = 2$. In the piecewise expression of $f(x)$, it is $\leq$ in the first part. Therefore,

$$f(2) = \lim_{x \to 2^-} f(x) = \lim_{x \to 2^-} \frac{x^2 - 3k}{x - 3} = \frac{4 - 3k}{-1} = 3k - 4$$

Similarly, we have

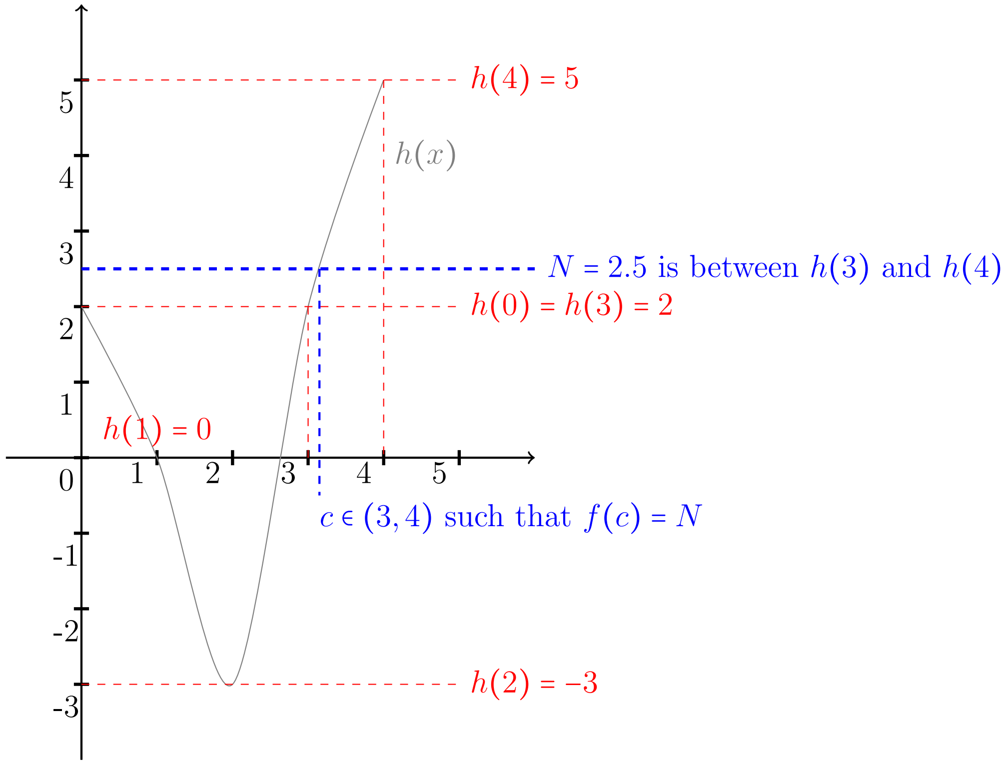
$$\lim_{x \to 2^-} f(x) = \lim_{x \to 2^+} 8x - k = 8 \cdot 2 - k = 16 - k$$

Now due to $(*)$, it is enough to let $3k - 4 = 16 - k$ and solve for k. □

Q6[*Sec1.8, Intermediate Value Theorem(IVT)*] Suppose function $h(x)$ is continuous on $[0,4]$. Suppose $h(0) = 2, h(1) = 0, h(2) = -3, h(3) = 2, h(4) = 5$. For what value of N, the must be a $c \in (3,4)$ such that $h(c) = N$?

Options: **A**. $N = 0.5$; **B**. $N = 0$; **C**. $N = -2$; **D**. $N = 2.5$.

Intermediate Value Theorem(IVT): If f is continuous on $[a,b], f(a) \neq f(b)$, and N is between $f(a)$ and $f(b)$ then there exists $c \in (a,b)$ that satisfies $f(c) = N$.

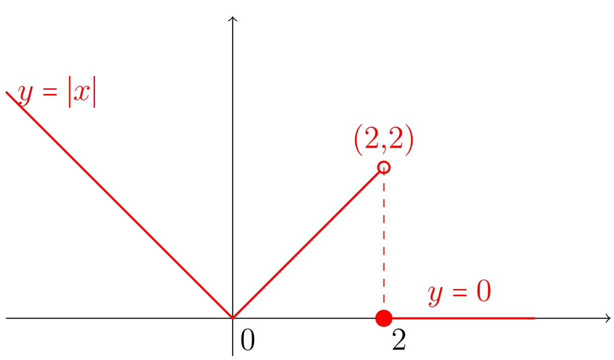


Q6[*Sec2.1/2.2, derivative at given point*] Select all true statements about the function $f(x) = \begin{cases} |x|, & x < 2 \\ 0, & x \geq 2 \end{cases}$

(False) I $f(x)$ is differentiable at $x = 0$.

(False) II $f(x)$ is continuous at $x = 2$

(True)III $\lim_{x\to 0} f(x)$ exists



Solution: From the above graph, $f(x)$ has a jump at $x = 2$ (the left and right parts are not connected), therefore, $f(x)$ is not continuous at $x = 2$. $f(x)$ has a sharp turn at $x = 0$, the left line has slope -1 and the right line has slope $+1$, therefore, $f(x)$ is not differentiable at $x = 0$. Also we can read the limits of f from the graph directly:

$$\lim_{0} f(x) = 0, \quad \lim_{2^-} f(x) = 2, \quad \lim_{2^+} f(x) = 0$$

Therefore, $\lim_{x\to 0} f(x)$ exists and $\lim_{x\to 2} f(x)$ does not exist. □

Q7[*Sec2.1/2.2, definition of derivative*] Let $f(x) = \frac{1}{x+1}$

(a) ***[Derivative as a limit]*** Use the definition of the derivative to find $f'(x)$. (Your calculation must include computing a limit.)

Solution:

$$f(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h\to 0} \frac{\frac{1}{x+h+1} - \frac{1}{x+1}}{h} = \lim_{h\to 0} \frac{\frac{1}{x+h+1} - \frac{1}{x+1}}{h} = \lim_{h\to 0} \frac{\frac{x+1}{(x+h+1)(x+1)} - \frac{x+h+1}{(x+1)(x+h+1)}}{h}$$

$$= \lim_{h\to 0} \frac{\frac{(x+1)-(x+h+1)}{(x+h+1)(x+1)}}{h}$$

$$= \lim_{h\to 0} \frac{\frac{-h}{(x+h+1)(x+1)}}{h}$$

$$= \lim_{h\to 0} \frac{-1}{(x+h+1)(x+1)}$$

$$= \frac{-1}{(x+0+1)(x+1)} = \frac{-1}{(x+1)^2}$$

□

(b) ***[Evaluating the derivative function at given point]*** Find $f'(2)$

Solution: $f'(2) = \frac{-1}{(2+1)^2} = -\frac{1}{9}$ □

(c) ***[Point-slope formula for the tangent line]*** Use part (b) to find an equation of a tangent line of $f(x)$ at $x = 2$.

Solution: Slope=$f'(2) = -\frac{1}{9}$. Point: $(2, f(2))$, where $f(2) = \frac{1}{2+1} = \frac{1}{3}$. According to the Point-Slope formula, the equation of the tangent line at $x = 2$ is given by:

$$y - \frac{1}{3} = (-\frac{1}{9})(x-2) \Longleftrightarrow y = (-\frac{1}{9})(x-2) + \frac{1}{3}$$

□

Q7*[*Sec2.1/2.2, definition of derivative*] Use the definition of the derivative to find $g'(1)$ for $g(x) = 2\sqrt{x}$.
Solution:

$$g'(1) = \lim_{h\to 0} \frac{g(1+h) - g(1)}{h} = \lim_{h\to 0} \frac{2\sqrt{1+h} - 2\sqrt{1}}{h} = \lim_{h\to 0} \frac{2\left(\sqrt{1+h} - 1\right)}{h} \cdot \frac{\sqrt{1+h}+1}{\sqrt{1+h}+1}$$

$$= \lim_{h\to 0} \frac{2\left(\sqrt{1+h} - 1\right)\left(\sqrt{1+h}+1\right)}{h\left(\sqrt{1+h}+1\right)}$$

$$= \lim_{h\to 0} \frac{2\left((\sqrt{1+h})^2 - 1^2\right)}{h\left(\sqrt{1+h}+1\right)}$$

$$= \lim_{h\to 0} \frac{2(1+h-1)}{h\left(\sqrt{1+h}+1\right)}$$

$$= \lim_{h\to 0} \frac{2h}{h\left(\sqrt{1+h}+1\right)}$$

$$= \lim_{h\to 0} \frac{2}{\sqrt{1+h}+1} = \frac{2}{\sqrt{1+0}+1} = 1$$

□

Q8[*Sec2.3/2.4/2.5, Differentiation Formulas/Laws*] Find the derivatives of the following functions. Do not need to simplify.

(a)[Linear Rule+Power functions]

$T(x) = 2\sqrt{x} - \frac{1}{2\sqrt{x}}$

Solution:

$$\begin{aligned} T'(x) = \Big(2\sqrt{x} - \frac{1}{2\sqrt{x}}\Big)' &= \Big(2x^{1/2}\Big)' - \Big(\frac{1}{2}x^{-1/2}\Big)' \\ &= 2\cdot\frac{1}{2}x^{1/2-1} - \frac{1}{2}\cdot(-\frac{1}{2})x^{-1/2-1} \\ &= x^{-1/2} + \frac{1}{4}x^{-3/2} \end{aligned}$$

□

(b)[Product Rule+Power functions]

$g(t) = \Big(\frac{1}{t^5} - 2t\Big)\Big(\frac{1}{\sqrt{t}} + \pi\Big)$

Solution:

$$\begin{aligned} g'(t) = \Big((\frac{1}{t^5} - 2t)(\frac{1}{\sqrt{t}} + \pi)\Big)' &= \Big(\frac{1}{t^5} - 2t\Big)'\cdot\Big(\frac{1}{\sqrt{t}} + \pi\Big) + \Big(\frac{1}{t^5} - 2t\Big)\cdot\Big(\frac{1}{\sqrt{t}} + \pi\Big)' \\ &= \Big(t^{-5} - 2t\Big)'\cdot\Big(t^{-1/2} + \pi\Big) + \Big(t^{-5} - 2t\Big)\cdot\Big(t^{-1/2} + \pi\Big)' \\ &= \Big((t^{-5})' - (2t)'\Big)\cdot\Big(t^{-1/2} + \pi\Big) + \Big(t^{-5} - 2t\Big)\cdot\Big((t^{-1/2})' + (\pi)'\Big) \\ &= \Big((-5)t^{-5-1} - 2\Big)\cdot\Big(t^{-1/2} + \pi\Big) + \Big(t^{-5} - 2t\Big)\cdot\Big(-\frac{1}{2}t^{-1/2-1} + 0\Big) \end{aligned}$$

□

(c)[Trig functions+Chain Rule]

$y = \sin(x^2)$

Solution: Outer function: $\sin(\blacksquare)$; Inner function: x^2.

$$\begin{aligned} &outer' = (\sin(\blacksquare))' = \cos(\blacksquare) \to (\text{plug inner } x^2 \text{ in}) \to \cos(x^2); \\ &inner' = (x^2)' = 2x \\ &y' = (\sin(x^2))' = outer'(inner)\cdot inner' = \cos(x^2)\cdot(2x) \end{aligned}$$

□

(c*)[Trig functions+Chain Rule]

$y = \sin^2(x)$

Solution: Outer function: $\blacksquare^2$; Inner function: $\sin x$.

$$\begin{aligned} &outer' = (\blacksquare^2)' = 2\blacksquare \to (\text{plug inner } \sin x \text{ in}) \to 2\sin x; \\ &inner' = (\sin x)' = \cos x \\ &y' = (\sin^2(x))' = outer'(inner)\cdot inner' = 2\sin x\cdot\cos x \end{aligned}$$

□

(d)[Quotient Rule+Trig functions+Chain Rule]

$f(t) = \dfrac{3t}{\tan(t^2-1)}$

Solution: Apply quotient rule first with Numerator: $3t$; Denominator: $\tan(t^2-1)$.

$$\begin{aligned} f'(t) = \left(\frac{3t}{\tan(t^2-1)}\right)' &= \frac{(numerator)' \cdot denominator - numerator \cdot (denominator)'}{(denominator)^2} \\ &= \frac{(3t)' \cdot \tan(t^2-1) - 3t \cdot (\tan(t^2-1))'}{(\tan(t^2-1))^2} \\ &= \frac{3 \cdot \tan(t^2-1) - 3t \cdot (\tan(t^2-1))'}{(\tan(t^2-1))^2} \end{aligned}$$

To compute $(\tan(t^2-1))'$, we need chain rule with Outer function $\tan(\blacksquare)$ and Inner function t^2-1.

$$\begin{aligned} &outer' = (\tan(\blacksquare))' = \sec^2(\blacksquare) \to (\text{plug inner } t^2-1 \text{ in}) \to \sec^2(t^2-1); \\ &inner' = (t^2-1)' = 2t - 0 = 2t \\ &(\tan(t^2-1))' = outer'(inner) \cdot inner' = \sec^2(t^2-1) \cdot (2t) \end{aligned}$$

Plug $(\tan(t^2-1))'$ back to the quotient rule, we have

$$f'(t) = \left(\frac{3t}{\tan(t^2-1)}\right)' = \frac{3 \cdot \tan(t^2-1) - 3t \cdot (\tan(t^2-1))'}{(\tan(t^2-1))^2} = \frac{3 \cdot \tan(t^2-1) - 3t \cdot \sec^2(t^2-1) \cdot (2t)}{(\tan(t^2-1))^2}$$

□

(e)[Trig functions+Double Chain Rule]

$f(x) = -2\sec\big(\cos(x^2+x)\big)$

Solution: $f(x)$ is a composition of three functions: $-\sec(\blacksquare), \cos(\blacksquare)$ and x^2+x. We need to apply chain rule twice.

1st Chain rule: Outer function: $-\sec(\blacksquare)$; Inner function: $\cos(x^2+x)$.

$$\begin{aligned} &outer' = (-\sec(\blacksquare))' = -\sec(\blacksquare) \cdot \tan(\blacksquare) \\ &\quad (\text{plug inner } \cos(x^2+x) \text{ in}) \to -\sec(\cos(x^2+x)) \cdot \tan(\cos(x^2+x)); \\ &inner' = (\cos(x^2+x))' \\ (*): \quad &f'(x) = outer'(inner) \cdot inner' = -\sec(\cos(x^2+x)) \cdot \tan(\cos(x^2+x)) \cdot (\cos(x^2+x))' \end{aligned}$$

To compute $(\cos(x^2+x))'$, we need to apply the second chain rule with Outer function: $\cos(\blacksquare)$; Inner function: x^2+x.

$$\begin{aligned} &outer' = (\cos(\blacksquare))' = -\sin(\blacksquare) \to (\text{plug inner } x^2+x \text{ in}) \to -\sin(x^2+x); \\ &inner' = (x^2+x)' = (x^2)' + x' = 2x+1 \\ (**): \quad &(\cos(x^2+x))' = outer'(inner) \cdot inner' = -\sin(x^2+x) \cdot (2x+1) \end{aligned}$$

Plug $(**)$ into $(*)$, we have

$$\begin{aligned} f'(x) &= -\sec(\cos(x^2+x)) \cdot \tan(\cos(x^2+x)) \cdot (\cos(x^2+x))' \\ &= -\sec(\cos(x^2+x)) \cdot \tan(\cos(x^2+x)) \cdot \Big(-\sin(x^2+x) \cdot (2x+1)\Big) \\ &= \sec(\cos(x^2+x)) \cdot \tan(\cos(x^2+x)) \cdot \sin(x^2+x) \cdot (2x+1) \end{aligned}$$

□

Q9[*Sec2.7, Rates of Change/Functions of motion*] A particle moves according to the law of motion $s(t) = t^3 - 5t^2 + 6t$, where t is measured in seconds and s in feet

(a)[1.4, Average velocity] Find the average velocity over the interval $[0, 2]$.

Solution: Average velocity=Average rate of change of $s(t)$ over $[0, 2]$

$$v_{ave} = \frac{s(2) - s(0)}{2 - 0} = \frac{(2^3 - 5 \cdot 2^2 + 6 \cdot 2) - (0)}{2} = \frac{8 - 20 + 12}{2} = 0 \text{ ft/s}$$

□

(b)[Velocity and position] Find the velocity at time t.

Solution:

$$v(t) = s'(t) = (t^3 - 5t^2 + 6t)' = (t^3)' - (5t^2)' + (6t)' = 3t^2 - 5 \cdot 2t + 6 = 3t^2 - 10t + 6$$

□

(c)[Acceleration and velocity] What is the acceleration after 6 seconds?

Solution:

$$a(t) = v'(t) = (3t^2 - 10t + 6)' = (3t^2)' - (10t)' + (6)' = 3 \cdot 2t - 10 + 0 = 6t - 10$$
$$a(6) = 6 \cdot 6 - 10 = 26 \text{ ft/s}^2$$

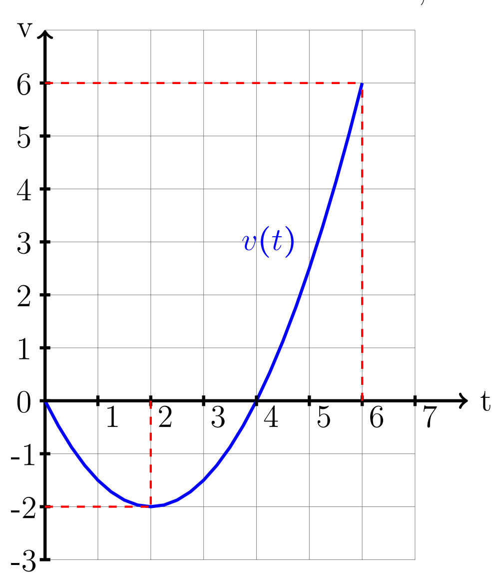
□

(d)[Velocity and speed] What is the speed of the particle when the acceleration is zero? **Solution:** $a(t) = 6t - 10 = 0 \Longrightarrow t = \frac{10}{6} = \frac{5}{3}$. Plug $t = 5/3$ into $v(t)$, we have

$$v(\frac{5}{3}) = 3(\frac{5}{3})^2 - 10\frac{5}{3} + 6 = -\frac{7}{3} \Longrightarrow \text{speed} = |v| = \frac{7}{3} \text{ ft/s}$$

□

Q10[*Sec2.7, Graph of the velocity*] The accompanying figure shows the velocity $v(t)$ of a particle moving on a horizontal coordinate line, for t in the closed interval $[0, 6]$.



Solution:

(a) When does the particle move forward?

Move forward $\Longleftrightarrow v > 0 \Longleftrightarrow t \in (4, 6)$

(b) When does the particle slow down?

Slow down $\Longleftrightarrow$ Speed $|v|$ drops $\Longleftrightarrow t \in (2, 4)$

(c) When is the particle's acceleration positive?

acceleration positive $\Longleftrightarrow a(t) = v'(t) > 0 \Longleftrightarrow$ slope of the tangent line is positive/v is increasing $\Longleftrightarrow t \in (2, 6)$

(d) When does the particle move at its greatest speed in $[0, 6]$?

greatest speed $\Longleftrightarrow$ highest or lowest point in the graph $\Longleftrightarrow t = 6$ (greatest speed=6)

Q11[*Sec2.6, Implicit differentiation*] Consider the curve $y^2 + 2xy + x^3 = x$

(a) Find the slope of the tangent line of the curve at the point $(1, -2)$.

Solution: Apply Implicit differential rule to the equation $y^2 + 2xy + x^3 = x$.

$$\begin{aligned} (y^2 + 2xy + x^3)' &= (x)' \\ \Longrightarrow (y^2)' + (2xy)' + (x^3)' &= 1 \qquad (*) \\ \Longrightarrow 2y \cdot y' + 2y + 2xy' + 3x^2 &= 1 \qquad (**) \end{aligned}$$

From $(*)$ to $(**)$, we use the chain rule for $(y^2)'$ and product rule for $(2xy)'$, where

$$\begin{aligned} &\text{chain rule}: (y^2)' = 2y(x) \cdot y'(x) = 2yy' \\ &\text{product rule}: (2xy)' = (2x)' \cdot y(x) + 2x \cdot y'(x) = 2y + 2xy' \\ &(x^3)' = 3x^2 \end{aligned}$$

Then plug $(x, y) = (1, -2), i.e., x = 1, y = -2$ into the above equation, we have

$$2 \cdot (-2) \cdot y' + 2(-2) + 2 \cdot 1 \cdot y' + 3 \cdot 1^2 = 1 \Longleftrightarrow -4y' - 4 + 2y' + 3 = 1 \Longleftrightarrow -2y' = 2 \Longleftrightarrow y' = -1$$

Therefore, the slope of the tangent line at $(1, -2)$ equals $y' = -1$ □

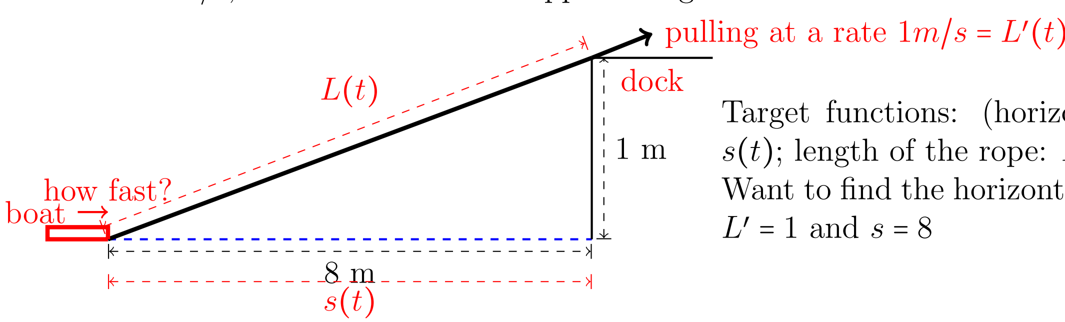
(b) Find the equation of the tangent line of the curve at the point $(1, -2)$.

Solution: Slope=-1. Point (1,-2). The slope-point formula gives the formula for the tangent line:

$$y - (-2) = (-1)(x - 1) \Longleftrightarrow y = (-1)(x - 1) - 2$$

□

Q12, *Sec2.8, Related Rates* A boat is pulled into a dock by a rope attached to the bow of the boat and passing through a pulley on the dock that is 1 m higher than the bow of the boat. If the rope is pulled in at a rate of 1 m/s, how fast is the boat approaching the dock when it is 8 m from the dock?



Target functions: (horizontal) position of the boat $s(t)$; length of the rope: $L(t)$.
Want to find the horizontal velocity $v(t) = s'(t)$ given $L' = 1$ and $s = 8$

Solution: Relation of s and L is given by Pythagorean theorem:

$$s^2(t) + 1^2 = L^2(t) \Longleftrightarrow s^2 + 1 = L^2 \qquad (*)$$

Take derivative both sides: $(s^2 + 1)' = (L^2)' \Longleftrightarrow (s^2)' + 1' = (L^2)'$. Since both $s = s(t)$ and $L = L(t)$ are functions, we need to apply chain rule to compute $(s^2)' = 2s \cdot s', (L^2)' = 2L \cdot L'$.

Now we get

$$2s \cdot s' + 0 = 2L \cdot L' \Longleftrightarrow s \cdot s' = L \cdot L' \qquad (**).$$

Give $s = 8$, from $(*)$ we can also figure out the corresponding L as $8^2 + 1 = L^2 \Longleftrightarrow L^2 = 61 \Longrightarrow L = \sqrt{65}$.
Now we can plug $s = 8, L = \sqrt{65}, L' = 1$ into $(**)$ to solve for s', i.e.,

$$8 \cdot s' = \sqrt{65} \cdot 1 \Longrightarrow s' = \frac{\sqrt{65}}{8} \Longrightarrow v(t) = s'(t) = \frac{\sqrt{65}}{8} \text{ m/s}$$

□