M L C — Solution.

Standard Response Questions. Show all work to receive credit. Please $\boxed{\textbf{BOX}}$ your final answer.

1. Given $f(x) = 2\cos(x) - \sin(2x)$

(a) (8 points) Find $f'(x)$.

$f'(x) = (2\cos x)' - (\sin 2x)'$

$= 2\cdot(-\sin x) - [\cos(2x)\cdot(2x)']$

$= -2\sin x - [\cos(2x)\cdot 2]$

$= \boxed{-2\cdot\sin x - 2\cos(2x)}$

(b) (5 points) Find the slope of the tangent line to $f(x)$ at the point $x = \pi$.

slope $= f'(\pi) = -2\cdot\sin\pi - 2\cdot\cos(2\pi)$ $\sin\pi = 0$

$= -2\cdot 0 - 2\cdot 1$ $\cos 2\pi = 1$

$= 0 - 2 = \boxed{-2}$

(c) (5 points) Find the equation of the tangent line to $f(x)$ at the point where $x = \pi$.

Tangent line equation: $y = f'(a)\cdot(x-a) + f(a)$

$a = \pi$, slope $= f'(\pi) = -2$.

$f(\pi) = 2\cos\pi - \sin(2\pi) = 2\cdot(-1) - 0$. $\sin 2\pi = 0$, $\cos\pi = -1$.

$= -2$

equation: $\boxed{y = -2\cdot(x-\pi) + (-2)}$

2. (a) (12 points) Use the definition of the derivative to compute the derivative of $f(x) = 5x + \sqrt{x}$.

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h\to 0} \frac{5(x+h) + \sqrt{x+h} - (5x+\sqrt{x})}{h}$$

$$= \lim_{h\to 0} \frac{\cancel{5x} + 5h + \sqrt{x+h} - \cancel{5x} - \sqrt{x}}{h}$$

$$= \lim_{h\to 0} \frac{5h + \sqrt{x+h} - \sqrt{x}}{h}$$

$$= \lim_{h\to 0} \frac{5\cancel{h}}{\cancel{h}} + \frac{\sqrt{x+h} - \sqrt{x}}{h}$$

$$= \lim_{h\to 0} 5 + \frac{(\sqrt{x+h} - \sqrt{x})}{h} \cdot \frac{(\sqrt{x+h} + \sqrt{x})}{\sqrt{x+h} + \sqrt{x}}$$

$$= \lim_{h\to 0} 5 + \frac{(\sqrt{x+h})^2 - (\sqrt{x})^2}{h(\sqrt{x+h} + \sqrt{x})}$$

$$= \lim_{h\to 0} 5 + \frac{\cancel{x} + \cancel{h} - \cancel{x}}{\cancel{h} \cdot (\sqrt{x+h} + \sqrt{x})}$$

$$= \lim_{h\to 0} 5 + \frac{1}{\sqrt{x+h} + \sqrt{x}}$$

$$= 5 + \frac{1}{\sqrt{x+0} + \sqrt{x}}.$$

$$= \boxed{5 + \frac{1}{2\sqrt{x}}}$$

(b) (6 points) Use your answer in part (a) to find an equation for the tangent line to $f(x)$ when $x = 4$.

slope $= f'(4) = 5 + \frac{1}{2\cdot\sqrt{4}} = 5 + \frac{1}{2\cdot 2} = 5 + \frac{1}{4} = \frac{21}{4}$

point : $(4, f(4))$. $f(4) = 5\cdot 4 + \sqrt{4} = 20 + 2 = 22$.

$(4, 22)$

Point-slope formula: $\boxed{y = \frac{21}{4}\cdot(x-4) + 22}$

3. At time t, the position of an object is given by the function $s = \frac{t^3}{3} - 2t^2 + 3t - 10$.

(a) (6 points) Find the object's acceleration each time the velocity is zero.

$v(t) = s'(t) = \left(\frac{t^3}{3} - 2t^2 + 3t - 10\right)' = \frac{1}{3}3\cdot t^2 - 2\cdot 2\cdot t + 3 - 0 = \boxed{t^2 - 4t + 3}$

$a(t) = v'(t) = (t^2 - 4t + 3)' = \boxed{2t - 4}$

$v(t) = t^2 - 4t + 3 = 0 \Leftrightarrow (t-1)(t-3) = 0 \Leftrightarrow t=1, t=3$.

$t=1$, $a(1) = 2\cdot 1 - 4 = \boxed{-2}$

$t=3$, $a(3) = 2\cdot 3 - 4 = \boxed{2}$

(b) (6 points) Find the object's velocity each time the acceleration is zero.

$a(t) = 2t - 4 = 0 \Rightarrow t = 2.$

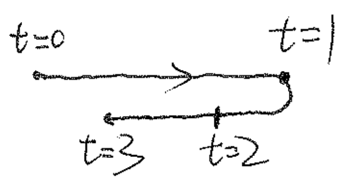
$\Rightarrow V(2) = 2^2 - 4\cdot 2 + 3 = 4 - 8 + 3 = \boxed{-1.}$

☆☆ (c) (6 points) Find the total distance travelled by the object from $t = 0$ to $t = 2$.

Notice $V(t) = t^2 - 4t + 3 = (t-1)\cdot(t-3)$

$0 < t < 1$, $V(t) > 0$, moving forward

$1 < t < 3$, $V(t) < 0$, moving backford



Therefore, the distance travelled from $t=0$ to $t=2$ is

$S(0) = 0 - 10 = -10$, $S(1) = \frac{1}{3} - 2 + 3 - 10 = -\frac{26}{3}$, $S(2) = \frac{8}{3} - 2\cdot 2^2 + 3\cdot 2 - 10 = -\frac{28}{3}$

Distance $= S(1) - S(0) + S(1) - S(2)$

$= \left(-\frac{26}{3} + 10\right) + \left(-\frac{26}{3} + \frac{28}{3}\right) = \frac{4}{3} + \frac{2}{3} = \boxed{2}$

4. Differentiate the following functions. **You need not simplify your answers.**

(a) (6 points) $y = 3x^4 + 2x\sin(x)$

$$y' = (3\cdot x^4)' + (2x\cdot \sin x)'$$
$$= 3\cdot 4\cdot x^3 + \underline{(2x)'\cdot \sin x + 2x\cdot(\sin x)'}$$
$$= \boxed{12\cdot x^3 + 2\cdot \sin x + 2x\cdot \cos x}$$

(b) (6 points) $f(x) = \dfrac{\sqrt{x}}{1+x^2}$

$$f'(x) = \frac{(\sqrt{x})'\cdot(1+x^2) - \sqrt{x}\,(1+x^2)'}{(1+x^2)^2}$$

$\sqrt{x} = x^{\frac{1}{2}}$, $(\sqrt{x})' = (x^{\frac{1}{2}})' = \frac{1}{2}x^{\frac{1}{2}-1}$

$(1+x^2)' = 1' + (x^2)' = 0 + 2x$.

$$= \boxed{\frac{\frac{1}{2}\cdot x^{\frac{1}{2}-1}\cdot(1+x^2) - \sqrt{x}\cdot 2x}{(1+x^2)^2}}$$

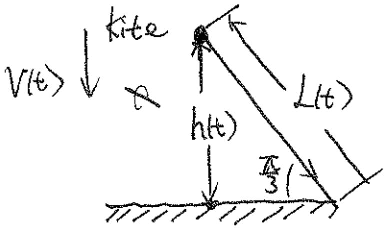
(c) (6 points) $r(\theta) = \sqrt{2\theta + \sin(5\theta)}$

Outer: $\sqrt{\blacksquare}$ Outer$' = (\sqrt{\blacksquare})' = (\blacksquare^{\frac{1}{2}})' = \frac{1}{2}\cdot \blacksquare^{-\frac{1}{2}} \xrightarrow[2\theta+\sin(5\theta)]{\text{Plug in}} \frac{1}{2}(2\theta+\sin(5\theta))^{-\frac{1}{2}}$

Inner: $2\theta + \sin(5\theta)$, Inner$' = 2 + (\sin(5\theta))' = 2 + \cos(5\theta)\cdot 5$

$$r'(\theta) = \boxed{\frac{1}{2}\cdot(2\theta + \sin(5\theta))^{-\frac{1}{2}}\cdot[2 + \cos(5\theta)\cdot 5]}$$

5. (18 points) A kite is flying at an angle of elevation of $\frac{\pi}{3}$ radians. The kite string is being taken in at the rate of 1 foot per second. If the angle of elevation does not change, how fast is the kite losing altitude?



rate: 1 ft/s $\Leftrightarrow L'(t) = 1$.

Target functions: string's length $L(t)$, rate $L'(t) = 1$

kite's height $h(t)$, vertical velocity $h'(t)$ (to be found)

Relation: $h(t) = L(t) \cdot \sin\frac{\pi}{3}$ (Right triangle)

$\Leftrightarrow h(t) = L(t) \cdot \frac{\sqrt{3}}{2} = \frac{\sqrt{3}}{2} \cdot L(t)$

Take derivative: $h'(t) = \frac{\sqrt{3}}{2} \cdot L'(t)$

Plug in $L'(t) = 1$

$\Rightarrow h'(t) = \frac{\sqrt{3}}{2} \cdot 1 = \boxed{\frac{\sqrt{3}}{2} \text{ ft/s.}}$

Multiple Choice. Circle the best answer. No work needed. No partial credit available.

6. (7 points) Find c such that the function $f(x) = \begin{cases} x^2 - 10, & \text{if } x \le c \\ 10x - 35, & \text{if } x > c \end{cases}$ is continuous everywhere.

A. $\sqrt{10}$
B. 5
C. 10
D. 20
E. None of the above.

only point in issue is $X=C$.

Plug in: $c^2-10 = 10\cdot c - 35$.

$\Leftrightarrow \; c^2-10c+25=0$

$\Leftrightarrow \; (c-5)^2=0$

$\Leftrightarrow \; c=5$.

7. (7 points) $f(x) = \dfrac{x^2-9}{x^2-9x+18}$ is continuous on:

A. $(-\infty,-3)\cup(-3,3)\cup(3,6)\cup(6,\infty)$
B. $(-\infty,6)\cup(6,\infty)$
C. $(-\infty,3)\cup(3,\infty)$
D. $(-\infty,3)\cup(3,6)\cup(6,\infty)$
E. None of the above.

$f(x) = \dfrac{x^2-9}{x^2-9x+18}$ is continuous on its domain

Denominator $\neq$ Zero.

$\Leftrightarrow \; x^2-9x+18 \neq 0$

$\Leftrightarrow \; (x-3)\cdot(x-6) \neq 0$

$\Leftrightarrow \; x\neq 3$ and $x \neq 6$.

f(x) is continuous everywhere

except 3 and 6.

8. (7 points) Consider proving that $\lim_{x\to 3}(2x-5) = 1$ using the formal definition of the limit. The largest value of $\delta > 0$ so that the formal definition of the limit holds is:

A. $\delta = \frac{\epsilon}{5}$
B. $\delta = \frac{\epsilon}{2}$
C. $\delta = 2\epsilon$
D. $\delta = 5\epsilon$
E. None of the above.

$a=3$. f(x) $L=1$.

$|f(x)-1| < \varepsilon$

$\Leftrightarrow |2x-5-1| < \varepsilon$

$\Leftrightarrow |2x-6| < \varepsilon$

$\Leftrightarrow |2(x-3)| < \varepsilon$

$\Leftrightarrow 2|x-3| < \varepsilon$

$\Leftrightarrow |x-3| < \frac{\varepsilon}{2}$

a δ

$\Rightarrow \boxed{\delta = \frac{\varepsilon}{2}}$

9. (7 points) Find the following limit: $\lim_{x\to -2} \frac{-x+3}{x^2+2x-15}$

A. $-\frac{1}{5}$

B. $-\frac{1}{3}$

C. $\frac{1}{3}$

D. $\frac{1}{5}$

E. None of the above.

Plug in -2

$$= \frac{-(-2)+3}{(-2)^2+2(-2)-15} = \frac{4+3}{4-4-15} = \boxed{\frac{7}{-15}}$$

10. (7 points) Find the following limit: $\lim_{x\to -3} \frac{x^2-9}{x^2+2x-3}$

$$= \frac{(-3)^2-9}{(-3)^2+2(-3)-3} = \frac{0}{0}$$

A. 0

B. $\frac{3}{2}$

C. 3

D. ∞

E. None of the above.

Plug -3 in

↳ Factorize

$$\frac{(x-3)\cancel{(x+3)}}{\cancel{(x+3)}\cdot(x-1)} = \cancel{\frac{x+3}{x+1}} \quad \frac{x-3}{x-1}$$

$$\lim_{x\to -3} \frac{x-3}{x-1} \xrightarrow{\text{Plug in}} \frac{(-3)-3}{(-3)-1} = \frac{-6}{-4} = \boxed{\frac{3}{2}}$$

☆☆☆11. (7 points) Find the following limit: $\lim_{x\to 9} \frac{\sin(\sqrt{x}-3)}{x-9}$

$$= \lim_{x\to 9} \frac{\sin(\sqrt{x}-3)}{\sqrt{x}-3} \cdot \frac{\sqrt{x}-3}{x-9}$$

$$= 1 \cdot \lim_{x\to 9} \frac{\cancel{\sqrt{x}-3}}{(\sqrt{x}+3)\cdot\cancel{(\sqrt{x}-3)}}$$

$$= 1 \cdot \lim_{x\to 9} \frac{1}{\sqrt{x}+3}$$

$$= \frac{1}{\sqrt{9}+3} = \frac{1}{3+3} = \boxed{\frac{1}{6}}$$

A. 0

B. $\frac{1}{3+\sqrt{3}}$

C. $\frac{1}{6}$

D. ∞

E. None of the above.

12. (7 points) Given the equation $x^2 + 2xy - y^2 + x = 2$, find $\dfrac{dy}{dx}$.

A. $4x + 1$

B. $\frac{-1-2x-2y}{2x-2y}$ ✓

C. $\frac{1-2x-2y}{2x-2y}$

D. $\frac{1-x-y}{y-x}$

E. None of the above.

$$(x^2 + 2x\cdot y - y^2 + x)' = 2'$$

$$\Leftrightarrow 2x + \underline{(2x)'\cdot y + 2x\cdot y'} - 2\cdot y\cdot y' + 1 = 0$$

$$\Leftrightarrow 2x + 2\cdot y + 2x\cdot y' - 2y\cdot y' + 1 = 0$$

$$(2x-2y)\cdot y' = -2x - 2y - 1$$

$$\Leftrightarrow \boxed{y' = \frac{-2x-2y-1}{2x-2y}}$$

13. (7 points) If the volume of a sphere is increasing at a constant rate of 6 m^3/s, how fast is its radius increasing when the radius is 3 m ?

A. 6π

B. $\frac{6}{\pi}$

C. $\frac{\pi}{6}$

D. $\frac{1}{6\pi}$ ✓

E. None of the above.

$V = \frac{4}{3}\pi r^3$, $V' = 6$.

$r = 3$.

$V' = \frac{4}{3}\pi\cdot 3\cdot r^2\cdot r'$

Plug in: $6 = \frac{4}{3}\pi\cdot 3\cdot 3^2\cdot r'$

Solve for r': $r' = \frac{6}{4\pi\cdot 9} = \frac{1}{6\pi}$

14. (7 points) On which interval(s) there must be a zero of the function $f(x) = x^3 - 5x - 2$?

A. Only $(-1, 0)$

B. Only $(0, 1)$

C. $(0, 1)$ and $(2, 3)$

D. $(-1, 0)$ and $(2, 3)$ ✓

E. None of the above.

$\Leftrightarrow$ On which interval(s) there must be a c s.t.

$f(c) = 0$ $(N=0)$

$f(-1) = -1 - 5\cdot(-1) - 2 = -1 + 5 - 2 = 2$

$f(0) = 0 - 0 - 2 = -2$

$f(1) = 1 - 5 - 2 = -6$

$f(2) = 2^3 - 5\cdot 2 - 2 = -4$

$f(3) = 3^3 - 5\cdot 3 - 2 = 10$