§2.3 Differential Formulas

- Basic Formulas: Notation: $f'(x) = \frac{df}{dx}$.

☆ Power function: For any n, $\boxed{(x^n)' = n \cdot x^{n-1}}$. In particularly, $(c)'=0$, $(x)'=1$.

eg.1. $(\sqrt[3]{x})' = (x^{\frac{1}{3}})' \overset{n=\frac{1}{3}}{=\!=} \frac{1}{3}\cdot x^{\frac{1}{3}-1} = \boxed{\frac{1}{3}\cdot x^{-\frac{2}{3}}}$

$(\frac{1}{x})' = (x^{-1})' \overset{n=-1}{=\!=} (-1)\cdot x^{-1-1} = \boxed{(-1)\cdot x^{-2} = \frac{-1}{x^2}}$

- Derivative Rules:

Sum/Difference: $[f(x) \pm g(x)]' = f'(x) \pm g'(x)$.

Constant multiple: $(c\cdot f(x))' = c\cdot(f(x))'$ for any constant c.

☆ Product: $[f(x)\cdot g(x)]' = f'(x)\cdot g(x) + f(x)\cdot g'(x)$. or $\boxed{(f\cdot g)' = f'g + f\cdot g'}$

☆ Quotient: $\left[\frac{f(x)}{g(x)}\right]' = \frac{f'(x)\cdot g(x) - f(x)\cdot g'(x)}{g^2(x)}$ or $\boxed{\left(\frac{f}{g}\right)' = \frac{f'g - f\cdot g'}{g^2}}$

eg.2. Let $f(x) = 5 - \frac{3}{\sqrt{x}} + 2\cdot x^{3.5}$. Compute $f'(x)$.

Solution: $f'(x) = 5' - \left(\frac{3}{\sqrt{x}}\right)' + (2\cdot x^{3.5})'$

$= 0 - 3\cdot(x^{-\frac{1}{2}})' + 2\cdot(x^{3.5})'$ Hint: $\frac{1}{\sqrt{x}} = \frac{1}{x^{\frac{1}{2}}} = x^{-\frac{1}{2}}$

$= 0 - 3\cdot(-\frac{1}{2})\cdot x^{-\frac{1}{2}-1} + 2\cdot 3.5\cdot x^{3.5-1} = \boxed{\frac{3}{2}\cdot x^{-\frac{3}{2}} + 7\cdot x^{2.5}}$

eg.3. (§16). Let $g(x) = (\frac{1}{x^5} - 2x)(\frac{1}{\sqrt{x}} + \pi)$. Compute $g'(x)$.

Solution: $g'(x) = \left[\frac{1}{x^5} - 2x\right]'(\frac{1}{\sqrt{x}} + \pi) + (\frac{1}{x^5} - 2x)(\frac{1}{\sqrt{x}} + \pi)'$

$= (x^{-5} - 2x)'\cdot(\frac{1}{\sqrt{x}} + \pi) + (\frac{1}{x^5} - 2x)\cdot(x^{-\frac{1}{2}} + \pi)'$

$= \boxed{(-5x^{-6} - 2)\cdot(\frac{1}{\sqrt{x}} + \pi) + (\frac{1}{x^5} - 2x)\cdot(-\frac{1}{2}\cdot x^{-\frac{3}{2}} + 0)}$

eg4. Let $f(x)=\frac{2}{x+1}$. Find the tangent line of $f(x)$ at $(3,\frac{1}{2})$.

Recall, the slope of the tangent line at $x=3$ equals the derivative at $x=3$.

i.e. We need to compute $f'(3)$ (or $\left.\frac{df}{dx}\right|_{x=3}$)

Solution: $f'(x)=\frac{2'(x+1)-2\cdot(x+1)'}{(x+1)^2}$ (quotient rule). $2'=0$, $(x+1)'=x'+1'=1+0$

$$=\frac{0-2}{(x+1)^2}=\frac{-2}{(x+1)^2}\Rightarrow f'(3)=\frac{-2}{(3+1)^2}=\frac{-2}{16}=-\frac{1}{8}$$

Therefore, the tangent formula:

$$\boxed{y-\frac{1}{2}=\left(-\frac{1}{8}\right)\cdot(x-3)}$$

Higher order derivatives:

The derivative function of $f'(x)$ is called the second derivative of $f(x)$, denoted

$$\boxed{f''(x)}=(f'(x))'=\frac{d^2f}{dx^2}=\frac{d}{dx}\left(\frac{df}{dx}\right)$$

In the same way, we can define third/fourth ... order derivative as $f'''(x)=[f''(x)]'$, ...

n-th order derivative is also denoted as $f^{(n)}(x)$, eg. $f^{(4)}(x)=f''''(x)=[f'''(x)]'$

Three important physical functions: Displacement $s(t)$, Velocity $V(t)$, Acceleration $a(t)$.

$\boxed{V(t)=s'(t)\ ,\ a(t)=V'(t)}$, i.e. $a(t)$ is the second order derivative of $s(t)$, i.e. $a(t)=s''(t)$.

e.g.5. A particle moves according $s(t)=t^3-6t^2+5\cancel{t}$, $t\geq 0$.
(F16). (a) Find the velocity at time t.

(b) What is the acceleration after 6 seconds?

Solution: (a) $V(t)=s'(t)=(t^3-6\cdot t^2+5\cancel{t})'=3\cdot t^2-6\cdot 2t+\overset{0}{\cancel{5}}=\boxed{3t^2-12t}$

(b) $a(t)=(V(t))'=(3t^2-12t)'=3\cdot 2t-12=\boxed{6t-12}$

so $a(6)=(6t-12)\Big|_{t=6}=6\cdot 6-12=\boxed{24}$ ft/s^2

§2.4 Trigonometric Derivatives

- Key formulas:
  - $(\sin x)' = \cos x$, $(\cos x)' = -\sin x$, $(\tan x)' = \sec^2 x$, $(\sec x)' = \tan x \sec x$
  - $\lim\limits_{\theta \to 0} \dfrac{\sin\theta}{\theta} = 1$.

eg1. Let $f(x) = 2\sin x - x\cdot\cos x$. Find $f'(x)$, $f''(x)$ and $f''(\frac{\pi}{2})$

solution:
$$\begin{aligned} f'(x) &= (2\sin x - x\cdot\cos x)' \\ &= (2\sin x)' - (x\cdot\cos x)' \\ &= 2(\sin x)' - (x\cdot\cos x)' \\ &= 2\cos x - (\cos x + x\cdot(-\sin x)) \\ &= \boxed{2\cos x - \cos x + x\cdot\sin x} = \boxed{\cos x + x\cdot\sin x} \end{aligned}$$

$(\sin x)' = \cos x$.

$(x\cdot\cos x)' \overset{\text{Product Rule}}{=} x'\cdot\cos x + x\cdot(\cos x)'$
$= 1\cdot\cos x + x\cdot(-\sin x)$

$$\begin{aligned} f''(x) = (f'(x))' &= (\cos x + x\cdot\sin x)' \\ &= (\cos x)' + (x\cdot\sin x)' \\ &= -\sin x + x'\cdot\sin x + x(\sin x)' \\ &= -\sin x + 1\cdot\sin x + x\cdot\cos x = \boxed{x\cdot\cos x} \end{aligned}$$

$$f''(\tfrac{\pi}{2}) = \tfrac{\pi}{2}\cos\tfrac{\pi}{2} = \tfrac{\pi}{2}\cdot 0 = \boxed{0}$$

eg.2 Let $h(\theta) = \dfrac{2\theta}{\sec\theta}$. Find $h'(\theta)$ and $h'(\frac{\pi}{4})$

$$\begin{aligned} h'(\theta) = \frac{(2\theta)'\cdot\sec\theta - (2\theta)\cdot(\sec\theta)'}{(\sec\theta)^2} &= \frac{2\cdot\sec\theta - 2\theta\cdot\tan\theta\sec\theta}{(\sec\theta)^2} \\ &= \frac{2\sec\theta\cdot[1-\theta\cdot\tan\theta]}{(\sec\theta)^2} \\ &= \frac{2(1-\theta\cdot\tan\theta)}{\sec\theta}. \end{aligned}$$

Hint: $(2\theta)' = 2$.
$(\sec\theta)' = \tan\theta\cdot\sec\theta$

$$h'(\tfrac{\pi}{4}) = \frac{2(1-\frac{\pi}{4}\cdot\tan\frac{\pi}{4})}{\sec\frac{\pi}{4}} = \boxed{\frac{2(1-\frac{\pi}{4}\cdot 1)}{\sqrt{2}}}$$

Hint: $\tan\frac{\pi}{4} = 1$.
$\sec\frac{\pi}{4} = \dfrac{1}{\cos\frac{\pi}{4}} = \sqrt{2}$.

eg3. Let $y(x) = \csc(x)$. Find $\left.\frac{dy}{dx}\right|_{x=\frac{\pi}{6}}$ and the tangent line at $x=\frac{\pi}{6}$.

Rmk: $(\csc x)' = -\csc x \cdot \cot x$ is in the formula sheet. (Not required).

$$\frac{dy}{dx} = -\csc x \cdot \cot x, \quad \left.\frac{dy}{dx}\right|_{x=\frac{\pi}{6}} = -\csc\frac{\pi}{6} \cdot \cot\frac{\pi}{6} = -2\cdot\sqrt{3}$$

Tangent line: slope $= -2\sqrt{3}$, Point: $\left(\frac{\pi}{6}, \csc\frac{\pi}{6}\right) = \left(\frac{\pi}{6}, 2\right)$

$$\boxed{y-2 = -2\sqrt{3}\cdot\left(x-\frac{\pi}{6}\right)}$$

- Application of the formula: $\lim_{\blacksquare \to 0} \frac{\sin \blacksquare}{\blacksquare} = 1$

eg4. Evaluate $\lim_{x\to 0} \frac{\sin(3x)}{2x}$. Rmk: It is $\frac{0}{0}$ type if you plug in 0.

Solution: $$\lim_{x\to 0}\frac{\sin(3x)}{2x} = \lim_{x\to 0}\frac{\sin(3x)}{2x}\cdot\frac{3x}{3x} = \lim_{x\to 0}\frac{\sin(3x)}{3x}\cdot\frac{3\cancel{x}}{2\cancel{x}}$$

$$= \left[\lim_{x\to 0}\frac{\sin(3x)}{3x}\right]\cdot\frac{3}{2}$$

$$= 1\cdot\frac{3}{2} = \boxed{\frac{3}{2}}$$

☆ eg5. Find the limit $\lim_{x\to 1}\frac{\sin(x-1)}{x^2-1}$

Solution: $$\lim_{x\to 1}\frac{\sin(x-1)}{x^2-1} = \lim_{x\to 1}\frac{\sin(x-1)}{x-1}\cdot\frac{x-1}{x^2-1}$$

$$= \left[\lim_{x\to 1}\frac{\sin(x-1)}{x-1}\right]\cdot\left[\lim_{x\to 1}\frac{x-1}{x^2-1}\right]$$

$$= 1 \cdot \lim_{x\to 1}\frac{\cancel{x-1}}{(x+1)\cancel{(x-1)}}$$

$$= 1 \cdot \frac{1}{1+1} = \boxed{\frac{1}{2}}$$

§2.5 Chain rule.

Key formula: $[f(g(x))]' = f'[g(x)] \cdot g'(x)$. Derivative of composition functions.

In Leibniz notation: $\dfrac{df}{dx} = \dfrac{df}{dg} \cdot \dfrac{dg}{dx}$.

Rmk1: $f(g(x))$ is called the composition of f following g. $f(x)$: Outer function, $g(x)$: Inner function.

e.g.1 $y=\sqrt{x^2+1}$ can be re-written as the composition of $f(x)=\sqrt{x}$ following $g(x)=x^2+1$

as $y=\sqrt{x^2+1} = f(g(x))$. Outer function $\sqrt{\blacksquare}$; Inner function: $\blacksquare^2+1$.

Rmk2: $f'[g(x)]$ means: find $f'(x)$ first, then plug in $g(x)$.

eg.2 Find y' of $y=\sqrt{x^2+1}$.

Solution: Step 1: $f(x)=\sqrt{x}$, $g(x)=x^2+1$

Step 2.1: $f'(x) = (x^{\frac{1}{2}})' = \frac{1}{2} \cdot x^{\frac{1}{2}-1} = \frac{1}{2} \cdot x^{-\frac{1}{2}}$

Step 2.2: Plug in $g(x)$, $f'[g(x)] = \boxed{\frac{1}{2}(x^2+1)^{-\frac{1}{2}}}$

Step 3: $g'(x) = (x^2+1)' = 2x + 0 = \boxed{2x}$

Step 4: $y' = (\sqrt{x^2+1})' = f'[g(x)] \cdot g'(x) = \boxed{\frac{1}{2} \cdot (x^2+1)^{-\frac{1}{2}} \cdot 2x} = (x^2+1)^{-\frac{1}{2}} \cdot x$.

eg.3 Let $f(t) = \dfrac{\cos(2\sin t)}{3}$. Find $f'(t)$.

Solution: $f(t) = \underbrace{\tfrac{1}{3}\cos}_{\text{outer}}[\underbrace{2\sin t}_{\text{inner}}]$

Derivative of outer: $(\frac{1}{3}\cos \blacksquare)' = \frac{1}{3}[\cos \blacksquare]' = \frac{1}{3}(-\sin \blacksquare) = -\frac{1}{3}\sin \blacksquare$

Plug in inner: $-\frac{1}{3}\sin(2\sin t)$

Derivative of inner: $(2\sin t)' = 2(\sin t)' = 2\cos t$.

Chain rule: $f'(t) = [\frac{1}{3}\cos(2\sin t)]' = \boxed{-\frac{1}{3}\sin(2\sin t) \cdot 2\cos t} = -\frac{2}{3}\sin(2\sin t) \cdot \cos t$.

eg.4. Let $y = \underbrace{\tan}_{\text{outer}}\underbrace{(3x)}_{\text{inner}}$ Find $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$

$(\tan \blacksquare)' = \sec^2(\blacksquare) \xrightarrow[3x]{\text{Plug in inner}} \sec^2(3x)$

$(3x)' = 3$

$\Rightarrow \frac{dy}{dx} = (\tan(3x))' = \boxed{\sec^2(3x) \cdot 3}$

$\frac{dy}{dx} = y' = 3 \cdot \sec^2(3x)$ $\quad \frac{d^2y}{dx^2} = y'' = (y')' = (3\sec^2(3x))' = \boxed{18 \cdot \sec^2(3x) \cdot \tan(3x)}$

$= 3[\sec(3x)]^2$ Outer: $3[\blacksquare]^2$. Inner: $\sec(3x)$.

(1st chain): $(\text{outer})' = (3\blacksquare^2)' = 3 \cdot 2 \cdot \blacksquare \xrightarrow[\sec(3x)]{\text{Plug in Inner}} 6 \cdot \sec(3x)$

$(\text{inner})' = (\underbrace{\sec}_{\text{outer}}\underbrace{(3x)}_{\text{inner}})' = [\tan(3x) \cdot \sec(3x)] \cdot 3$

$\Rightarrow (3\sec^2(3x))' = [6\sec(3x)] \cdot [\tan(3x)\sec(3x) \cdot 3]$

$= 18 \cdot \sec^2(3x) \cdot \tan(3x)$.

(2nd chain): $(\sec \bullet)' = \tan \bullet \cdot \sec \bullet \xrightarrow{3x} \tan(3x)\sec(3x)$

$(3x)' = 3$.

Remark: It is actually the composition of three functions, which requires chain rule TWICE.

$$(f[g(h(x))])' = f'(g(h(x))) \cdot [g(h(x))]' = f'(g(h(x))) \cdot g'(h(x)) \cdot h'(x).$$

eg.5. (Product rule + Chain rule). Find the derivative of $f(x) = \frac{x}{\tan(x^2-1)}$.

(S16). $f'(x) = \frac{x' \cdot \tan(x^2-1) - x \cdot [\tan(x^2-1)]'}{[\tan(x^2-1)]^2} = \boxed{\frac{\tan(x^2-1) - x \cdot [\sec^2(x^2-1) \cdot 2x]}{[\tan(x^2-1)]^2}}$

$[\underbrace{\tan}_{\text{outer}}\underbrace{(x^2-1)}_{\text{inner}}]' = \underbrace{[\sec^2(x^2-1)]}_{(\text{outer})'} \cdot \underbrace{(x^2-1)'}_{(\text{inner})'} = \sec^2(x^2-1) \cdot 2x$.

eg.6. Let $h(x) = f(g(x))$ where $\boxed{g(1)=2}$, $\boxed{g'(1)=3}$, $f(1)=4$, $f'(1)=5$, $f(2)=6$, $\boxed{f'(2)=7}$

(S16, MC) Find $h'(1)$. Rmk: Compute $h'(x)$ then plug in $x=1$.

$h'(x) = [f(g(x))]' = f'(g(x)) \cdot g'(x)$

$\Rightarrow h'(1) = f'(g(1)) \cdot g'(1) = f'(2) \cdot 3 = 7 \cdot 3 = \boxed{21}$.