§2.1. Derivatives

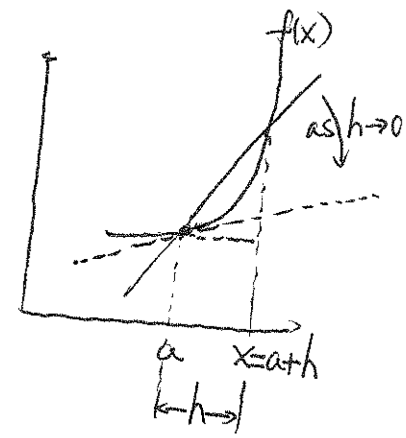
key points: ① Definition of Derivative: as a limit of average rate of change.

② Slope of tangent line as a derivative and the formula of tangent line.

③ Left and right derivatives of piecewise functions.

• Definition: The derivative of a function $f(x)$ at $x=a$, denoted $f'(a)$,

given by: $\boxed{f'(a) = \lim_{h\to 0} \frac{f(a+h)-f(a)}{h}}$



Remark 1: If we write $h = x - a \Leftrightarrow x = a+h$, then $h \to 0$ (h approaches 0) is equivalent to $x - a \to 0 \Leftrightarrow x \to a$ (x approaches a).

So $f'(a)$ can also be defined as $\boxed{f'(a) = \lim_{x\to a} \frac{f(x)-f(a)}{x-a}}$

Remark 2: Notice that the above ratios are actually the average rate of change of the function over the interval $[a, a+h] = [a, x]$, which is the slope of the secant line. And the limit will be the slope of the tangent line passing through $(a, f(a))$, i.e., the slope of the tangent line $= f'(a)$.

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e.g. 1: Let $f(x) = \frac{1}{x+1}$. Find $f'(2)$ and the formula of the tangent line through $(2, f(2))$.

Solution: $f'(2) = \lim_{h\to 0} \frac{f(2+h)-f(2)}{h} = \lim_{h\to 0} \frac{\frac{1}{2+h+1} - \frac{1}{2+1}}{h}$ (simplify)

$= \lim_{h\to 0} \frac{\frac{3-(3+h)}{(3+h)\cdot 3}}{h} = \lim_{h\to 0} \frac{-\cancel{h}}{(3+h)\cdot 3\cdot \cancel{h}}$ Hint: $\frac{\frac{a}{b}}{c} = \frac{a}{b\cdot c}$

Notice $f(2) = \frac{1}{2+1} = \frac{1}{3}$.

$= \lim_{h\to 0} \frac{-1}{(3+h)\cdot 3} = \boxed{\frac{-1}{9}}$ (Plug in $h=0$).

Tangent line: through $(2, \frac{1}{3})$ with slope $f'(2) = -\frac{1}{9}$.

has formula: $\boxed{y - \frac{1}{3} = -\frac{1}{9}\cdot(x-2)}$

eg 2. Find $f'(0)$ for $f(x)=\sqrt{1-x}$.

Solution: $f'(0)=\lim\limits_{h\to 0}\dfrac{f(0+h)-f(0)}{h}=\lim\limits_{h\to 0}\dfrac{\sqrt{1-(0+h)}-\sqrt{1-0}}{h}$

$=\lim\limits_{h\to 0}\dfrac{\sqrt{1-h}-1}{h}$

Notice that if we plug in $h=0$, we have $\frac{0}{0}$. So we need to simplify the numerator first.

☆ (Conjugation Method).

$\sqrt{1-h}-1=(\sqrt{1-h}-1)\times\dfrac{\sqrt{1-h}+1}{\sqrt{1-h}+1}$

$=\dfrac{(\sqrt{1-h}-1)(\sqrt{1-h}+1)}{\sqrt{1-h}+1}=\dfrac{(\sqrt{1-h})^2-1^2}{\sqrt{1-h}+1}$

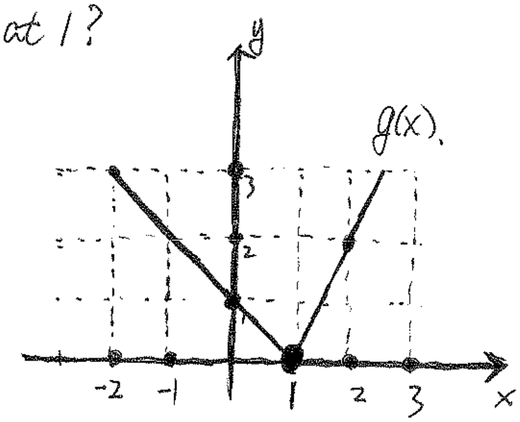
$=\dfrac{1-h-1}{\sqrt{1-h}+1}=\dfrac{-h}{\sqrt{1-h}+1}$

Remark: $\sqrt{A}+\sqrt{B}$ is called the conjugate radical of $\sqrt{A}-\sqrt{B}$. Notice that $(\sqrt{A}-\sqrt{B})(\sqrt{A}+\sqrt{B})=(\sqrt{A})^2-(\sqrt{B})^2=A-B$ helps remove the square root.

Therefore, $f'(0)=\lim\limits_{h\to 0}\dfrac{\frac{-h}{\sqrt{1-h}+1}}{h}=\lim\limits_{h\to 0}\dfrac{-1}{\sqrt{1-h}+1}=\boxed{\dfrac{-1}{2}}$ (Plug in $h=0$).

- $\lim\limits_{h\to 0^-}\dfrac{f(a+h)-f(a)}{h}$ and $\lim\limits_{h\to 0^+}\dfrac{f(a+h)-f(a)}{h}$ are called left and right derivatives of $f(x)$ at a.
- If $\lim\limits_{h\to 0}\dfrac{f(a+h)-f(a)}{h}$ does not exist, we say $f(x)$ does not derivative at a.
- Linear function $y=k\cdot x+b$ has derivative k at any point, since all the secant lines are the same. The tangent line is the line itself.

eg.3 Given the graph of $g(x)$. Does $g(x)$ has derivative at 1?



Solution: Based on the geometric meaning of derivative, the left and Right derivates of $g(x)$ at $x=1$ are exactly the two slopes the two straight lines, i.e. $\lim\limits_{h\to 0^-}\dfrac{g(1+h)-g(1)}{h}=-1$

$\lim\limits_{h\to 0^+}\dfrac{g(1+h)-g(1)}{h}=2$

Left and right limits (derivatives) are not the same, therefore, $\lim\limits_{h\to 0}\dfrac{g(1+h)-g(1)}{h}$ D.N.E. $g(x)$ does not have derivative at $x=1$.

§2.2 Derivative Function

Key points: ① Compute $f'(x)$ via limit definition of derivative.

② Graph $f'(x)$ based on the graph of $f(x)$.

Replace a by x in $f'(a)$ and consider it as a new function of x. i.e.

Def: $$f'(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}=\lim_{z\to x}\frac{f(z)-f(x)}{z-h}$$ is called the derivative function.

Remarks:
- The domain of $f'(x)$ is where $f(x)$ has derivative.
- The process of computing $f'(x)$ is also called differentiate $f(x)$.
- If $f'(a)$ exists (at $x=a$), we say $f(x)$ is differentiable at a.
- We also have the following notations for derivative:

$$f'(x)=\frac{df}{dx}, \quad f'(a)=\frac{df}{dx}\Big|_{x=a}$$

- Differentiable is stronger than continuous: If $f(x)$ is differentiable at a, then $f(x)$ is continuous at a, not vice versa.

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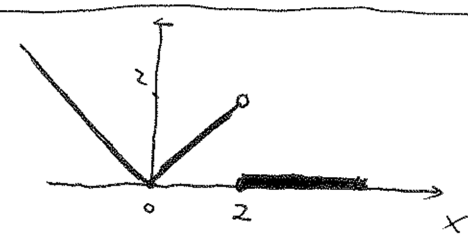
eg.1. True or Fasle: $f(x)=\begin{cases}|x| & \text{if } x<2\\ 0 & \text{if } x\geq 2\end{cases}$
(316)



(I) $f(x)$ is differentiable at $x=0$. (Fasle)

(II) $f(x)$ is continuous at $x=2$ (Fasle)

(III) $\lim_{x\to 0} f(x)$ exists.

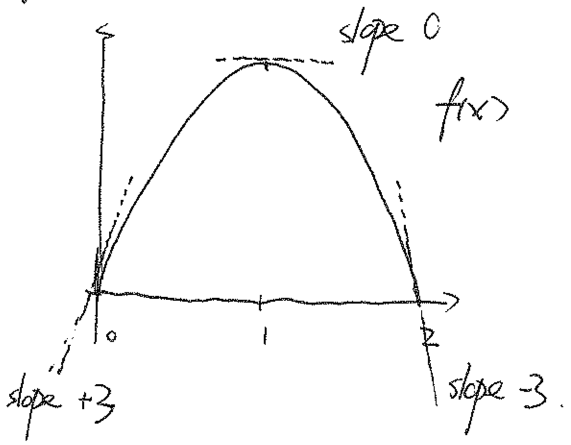
eg 2. Compute $r'(t)$ for $r(t)=3t+\frac{5}{t}$.

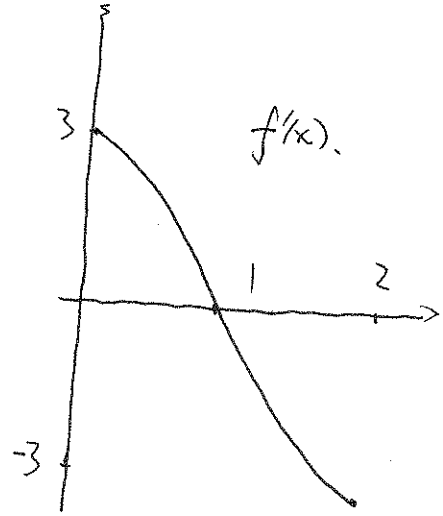
solution: $r'(t)=\lim_{h\to 0}\frac{r(t+h)-r(t)}{h}$ (think $r(t)$ as $f(x)$, where t plays the role of x)

$$=\lim_{h\to 0}\frac{\left[3(t+h)+\frac{5}{t+h}\right]-\left[3t+\frac{5}{t}\right]}{h}$$

$$=\lim_{h\to 0}\frac{\cancel{3t}+3h-\cancel{3t}+\frac{5}{t+h}-\frac{5}{t}}{h}=\lim_{h\to 0}\frac{3h}{h}+\frac{\frac{5t-5t-5h}{(t+h)\cdot t}}{h}=\lim_{h\to 0}3-\frac{5}{(t+h)\cdot t}=\boxed{3-\frac{5}{t^2}}$$

eg 3. For the function $f(x)$ shown below, sketch the graph of $f'(x)$.





eg 4 Consider $f(x)=\sqrt{1-2x}$. Find $f'(x)$ (via definition) and find an equation of a tangent line of $f(x)$ at $x=-4$.
(F16).

Solution: (a) $f'(x)=\lim\limits_{h\to 0}\dfrac{f(x+h)-f(x)}{h}=\lim\limits_{h\to 0}\dfrac{\sqrt{1-2(x+h)}-\sqrt{1-2x}}{h}$. (conjugation method in §2.1)

$$\sqrt{1-2(x+h)}-\sqrt{1-2x}=\frac{\left(\sqrt{1-2(x+h)}-\sqrt{1-2x}\right)\cdot\left(\sqrt{1-2(x+h)}+\sqrt{1-2x}\right)}{\sqrt{1-2(x+h)}+\sqrt{1-2x}}$$

$$=\frac{[1-2(x+h)]-[1-2x]}{\sqrt{1-2(x+h)}+\sqrt{1-2x}}=\frac{-2h}{\sqrt{1-2(x+h)}+\sqrt{1-2x}}.$$

Therefore, $f'(x)=\lim\limits_{h\to 0}\dfrac{\frac{-2h}{\sqrt{1-2(x+h)}+\sqrt{1-2x}}}{h}=\lim\limits_{h\to 0}\dfrac{-2}{\sqrt{1-2(x+h)}+\sqrt{1-2x}}\ \overset{\text{Plug in}}{\underset{h=0}{=}}\ \dfrac{-2}{\sqrt{1-2x}+\sqrt{1-2x}}$

$$=\boxed{\frac{-1}{\sqrt{1-2x}}}$$

(b). slope of the tangent line at $x=-4$:

$$f'(-4)=\frac{-1}{\sqrt{1-2(-4)}}=\frac{-1}{\sqrt{1+8}}=\frac{-1}{\sqrt{9}}=\frac{-1}{3}.$$

passing through the point $(-4, f(-4))=(-4, \sqrt{1-2\cdot(-4)})=(-4, 3)$.

Equation: $y-3=-\frac{1}{3}\cdot(x-(-4))$

$$\boxed{y-3=-\frac{1}{3}\cdot(x+4)}$$