# §3.9 Antiderivatives (Anti-D)

Key points: ① Definition of anti-D and the most general anti-D of $f(x)$.

② Finding anti-D using derivative table and linear rules.

③ Velocity and position as anti-D.

Definition: If $F'(x) = f(x)$, then $F(x)$ is ONE ANTIDERIVATIVE of $f(x)$.

$F(x)+C$ is called THE MOST GENERAL ANTIDERIVATIVE of $f(x)$, where $C$ is arbitrary constant.

e.g. 1. $(x^2)' = 2x$, $2x$ is the derivative of $x^2$; $x^2$ is one anti-D of $2x$.

$(x^2+5)' = 2x$, $2x$ is the derivative of $x^2+5$; $x^2+5$ is one anti-D of $2x$.

For any constant $C$, $(x^2+C)' = 2x$, $2x$ is the derivative of $x^2+C$, $x^2+C$ is one anti-D of $2x$.

$x^2+C$ is called the most general anti-D of $2x$.

• (Anti) derivative table:

| $F(x)$ | $f(x) = F'(x)$ | | | | |
|---|---|---|---|---|---|
| ☆ $x^n$ | $n \cdot x^{n-1}$ | $n \cdot x^{n-1}$ has anti-D $x^n$ | $\boxed{x^n \text{ has anti-D } \frac{1}{n+1} x^{n+1}}$ $(n \neq -1)$ |
| $\sin x$ | $\cos x$ | $\cos x$ has anti-D $\sin x$ | $\cos x$ has anti-D $\sin x$ |
| $\cos x$ | $-\sin x$ | $-\sin x$ has anti-D $\cos x$ | $\boxed{\sin x \text{ has anti-D } -\cos x}$ |
| $\tan x$ | $\sec^2 x$ | $\sec^2 x$ has anti-D $\tan x$ | |
| $\sec x$ | $\tan x \cdot \sec x$ | $\tan x \sec x$ has anti-D $\sec x$ | |

• Linear rule: If $f(x)$ has anti-D $F(x)$, $g(x)$ has anti-D $G(x)$, then

$a \cdot f(x) + b \cdot g(x)$ has anti-D $a F(x) + b \cdot G(x)$

eg. 2. Find one anti-D of (a) $f(x) = 2x^5$, (b) $f(x) = \frac{\sin x}{2}$, (c) $f(x) = 2x^5 + \frac{\sin x}{2}$.

(a): $f(x) = 2 \cdot \frac{1}{6} \cdot \boxed{6x^5}$. Notice $(x^6)' = 6 \cdot x^5 \Rightarrow$ anti-D of $f(x)$ is $F(x) = 2 \cdot \frac{1}{6} \cdot x^6$

(b): $f(x) = \frac{\sin x}{2} = \frac{(-1)}{2} \cdot (-\sin x)$. $(\cos x)' = -\sin x \Rightarrow$ anti-D of $f(x)$ is $F(x) = \frac{-1}{2} \cdot \cos x$.

(c) According to (a),(b). $2x^5 + \frac{\sin x}{2}$ has one anti-D $2 \cdot \frac{1}{6} x^6 + \frac{-1}{2} \cos x$.

☆ Key anti-D formula: $\boxed{x^n \xrightarrow{\text{anti-D}} \frac{1}{n+1}x^{n+1}}$, $n \neq -1$

eg 3. Find one anti-D F for the following functions f:

(a): $f(x)=1 \Rightarrow F(x)=x$ ; (a'): $f(x)=-\frac{1}{3} \Rightarrow F(x)=-\frac{1}{3}\cdot x$. (formula with $n=0$)

(b): $f(x)=5x^1 \Rightarrow F(x)=5\cdot\frac{1}{1+1}\cdot x^{1+1}=5\cdot\frac{1}{2}x^2$ (formula with $n=1$)

(c): $f(t)=t^3 \Rightarrow F(t)=\frac{1}{3+1}\cdot t^{3+1}=\frac{1}{4}\cdot t^4$ (formula with $n=3$)

Remark: The formula is also applied to negative n and fraction n.

☆ (d): $f(x)=\frac{1}{x^2} \Rightarrow f(x)=x^{-2} \Rightarrow F(x)=\frac{1}{-2+1}\cdot x^{-2+1}=-x^{-1}=\frac{-1}{x}$ ($n=-2$)

☆ (e): $f(t)=2\sqrt{t} \Rightarrow f(t)=2\cdot t^{\frac{1}{2}} \Rightarrow F(t)=2\cdot\frac{1}{\frac{1}{2}+1}\cdot t^{\frac{1}{2}+1}=2\cdot\frac{1}{\frac{3}{2}}\cdot t^{\frac{3}{2}}=\frac{4}{3}\cdot t^{\frac{3}{2}}$ ($n=\frac{1}{2}$)

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☆ According to the definition of Anti-D, the (most general) anti-D of $f'(x)$ is $f(x)+C$. With extra condition on $f(x)$, we can determine the value of C.

eg 4. Suppose $f'(x)=\sin x$ and $f(\frac{\pi}{2})=0$. Find $f(x)$.

Solution: $f(x)$ is the anti-D of $f'(x)=\sin x$. Therefore, $f(x)=-\cos x+C$

Furthermore, plug $x=\frac{\pi}{2}$ into $f(x)=-\cos x+C$, we have,

$$f(\tfrac{\pi}{2})=-\cos\tfrac{\pi}{2}+C \iff 0=-0+C \quad \text{since } f(\tfrac{\pi}{2})=0,\ \cos\tfrac{\pi}{2}=0$$

$$\Rightarrow C=0 \quad \text{plug into } f(x)=-\cos x+C.$$

$$\boxed{f(x)=-\cos x}$$

Remark: When you get the expression for $f(x)$, it is convenient to double check your answer by computing $f'(x)$ and $f(\frac{\pi}{2})$.

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- Moving particle. Position: $s(t)$. Velocity: $v(t)$. Acceleration: $a(t)$

Relation: $s'(t)=v(t)$, $v'(t)=a(t)$

$s(t)$ is the anti-D of $v(t)$ ; $v(t)$ is the anti-D of $a(t)$

Related problems: Give $v(t)$, find $s(t)$. Give $a(t)$, find $v(t)$.

e.g. 5. A particle is moving along a line with acceleration given by $a(t)=4t^3+2\sin t$.
(f16). Given the initial ~~value~~ velocity is $V(0)=5$ m/s, find the velocity at time $t=\pi$ seconds.

Hint: $V$ is the (general) anti-D of $a(t)$. Find the general anti-D of $4t^3+2\sin t$. Then use the initial condition to determine the constant $C$

Solution: $t^3 \xrightarrow{\text{anti-D}} \frac{1}{3+1} t^{3+1} = \frac{1}{4} t^4$ $(n=3)$ ; $\sin t \xrightarrow{\text{anti-D}} -\cos t$.

The general anti-D of $a(t)=4t^3+2\sin t$ is $4\left(\frac{1}{4}t^4\right)+2(-\cos t)+C$

i.e. $\boxed{V(t)} = 4\left(\frac{1}{4}t^4\right)+2(-\cos t)+C = \boxed{t^4-2\cos t+C}$

Plug in $t=0$: $\quad 5=V(0)=0^4-2\cos 0+C = 0-2+C$ since $\cos 0=1$.

$\Rightarrow 5=-2+C \Rightarrow C=7$. plug back into $V$'s expression.

$$\boxed{V(t)=t^4-2\cos t+7}$$

Then evaluate $V$ at $t=\pi$, i.e., $\boxed{\begin{aligned} V(\pi) &= \pi^4-2\cdot\cos\pi+7 \\ &= \pi^4+9 \text{ m/s}\end{aligned}}$ ; $\cos\pi=-1$.

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Hints for WW.

※3. Rewrite $f(x)=\dfrac{7-5x^9}{x^3} = \dfrac{7}{x^3}-\dfrac{5x^9}{x^3} = 7\cdot x^{-3}-5\cdot x^6$

※4. $y=f(x)$ goes through $(1,0)$ means $f(1)=0$.

The slope of the tangent line $= f'(x)=\dfrac{6}{x^3}-\dfrac{9}{x^5} = 6x^{-3}-9\cdot x^{-5}$

Then use the method in e.g. 4 to find $f(x)$.

※5. $\dfrac{1}{\sqrt[3]{x}} = x^{-\frac{1}{3}}$, $\sqrt[3]{x^2}=x^{\frac{2}{3}}$. Apply the anti-D formula with $n=-\frac{1}{3}$ and $n=\frac{2}{3}$

※6. $(x+4)^n$ has anti-D $\dfrac{1}{n+1}(x+4)^{n+1}$. For example, $(x+4)^3 \xrightarrow{\text{anti-D}} \frac{1}{4}(x+4)^4$.

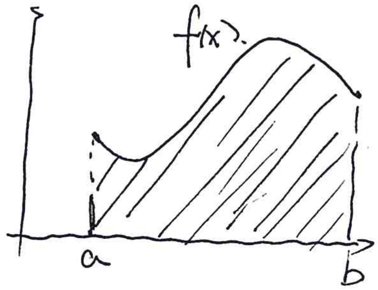
$(x+4)^4 \xrightarrow{\text{anti-D}} \frac{1}{5}(x+4)^5$

※7. $1 \text{ mph} = \dfrac{1 \text{ mile}}{1 \text{ hour}} = \dfrac{22}{15}$ ft/second.

Decelerate at 26 ft/s means $a(t)=-26$ ft/s² $\Rightarrow V(t)=-26\cdot t$ ft/s

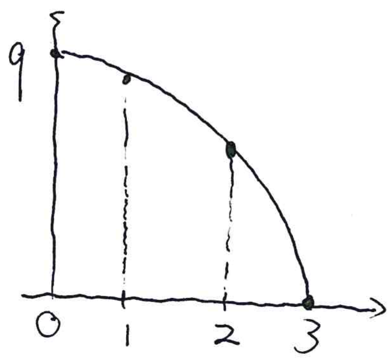
# §4.1 Area and distance.

- Goal: Use equally-spaced rectangles to estimate the area.




- left / Right endpoints sums
- Upper / Lower sums
  Over / Under - estimates

- eg1. Use three rectangles of equal width to estimate the area of $y = -x^2 + 9$ between $x=0$ and $x=3$, above x-axis.




| $x$ | 0 | 1 | 2 | 3 |
|---|---|---|---|---|
| $f(x)$ | 9 | 8 | 5 | 0 |

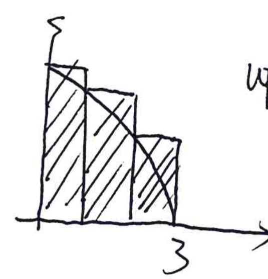
interval $[0,1], [1,2], [2,3]$

width. $\frac{3-0}{3} = 1$

Leftendpoints sum: $1\cdot f(0) + 1\cdot f(1) + 1\cdot f(2) = 9 + 8 + 5 = 22$

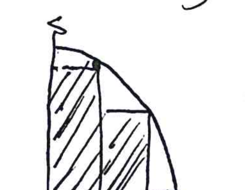
Right endpoints sum: $1\cdot f(1) + 1\cdot f(2) + 1\cdot f(3) = 8 + 5 + 0 = 13$

Upper sum (Over estimate) : $1\cdot f(0) + 1\cdot f(1) + 1\cdot f(2) = 22$




upper sum

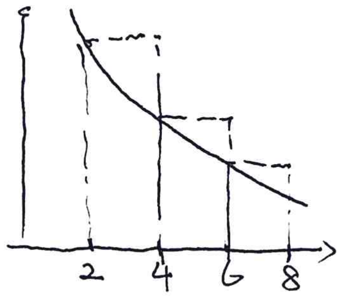
Lower sum (Under estimate) : $1\cdot f(1) + 1\cdot f(2) + 1\cdot f(3) = 13$



lower sum.

eg 2. Using three equally-spaced rectangles of equal width, find
(S17) the upper sum approximation of the area between the curve $y=\frac{1}{x}$ and the x-axis from $x=2$ to $x=8$.

sln:




Total interval: $[2,8]$. Total width: $8-2=6$

number of subinterval: 3. Width of each subinterval: $\frac{6}{3}=2$

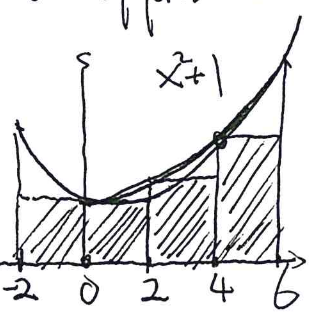
Three intervals: $[2,4]$, $[4,6]$, $[6,8]$

Upper Sum (= left endpoint in this case)

$$= 2\cdot(f(2)+f(4)+f(6)) = 2\left(\frac{1}{2}+\frac{1}{4}+\frac{1}{6}\right) = \boxed{\frac{11}{6}}$$

☆eg 3 Find the ~~too~~ under-estimate using 4 equally-spaced rectangles to approximate the area between $y=x^2+1$, x-axis from $x=-2$ to $x=6$

sln:




Underestimate: choose the rectangles BELOW the curve

1st interval: $[-2,0]$. height: $f(0)$.

2nd interval: $[0,2]$. height: $f(0)$

3rd interval: $[2,4]$ height: $f(2)$.

4th interval: $[4,6]$ height: $f(4)$

Lower sum: $2\cdot[f(0)+f(0)+f(2)+f(4)] = 2\cdot[1+1+5+17] = 48$

Remark: Overestimate: $2\cdot[f(-2)+f(2)+f(4)+f(6)] = 2\cdot[5+5+17+37]$

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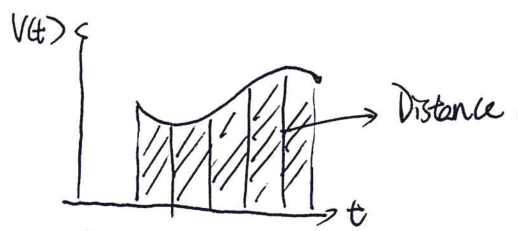
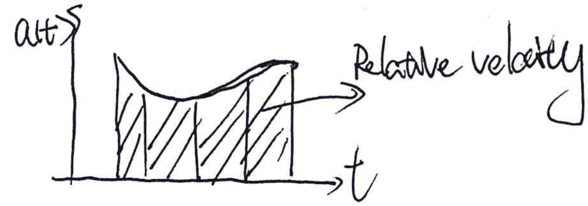
(Hints for webwork 4,5)

- Distance = Area below the velocity $v(t)$.




- Relative velocity = Area below acceleration $a(t)$



# §4.1. Part II. § Appendix E Sigma Notation

$$\overset{\text{end}\to}{} \quad \text{sum}\to \sum_{\underset{\text{index}\nearrow\ \ \uparrow\text{start}}{i=m}}^{n} a_i = a_m + a_{m+1} + a_{m+2} + \cdots + a_{n-2} + a_{n-1} + a_n$$

( $\Sigma$ read as Sigma ).

eg.1. $\sum_{i=1}^{5} 2i = 2\cdot 1 + 2\cdot 2 + 2\cdot 3 + 2\cdot 4 + 2\cdot 5$

eg.2. Write $1+2+4+8$ in the sigma notation. Hint: $1=2^0$, $2=2^1$, $4=2^2$, $8=2^3$

Formula for $i$th term: $2^i$

$i$ starts from 0, ends at 3.

$$1+2+4+8 = \sum_{i=0}^{3} 2^i$$

eg.3 Find the sum $\sum_{i=99}^{100} \frac{2}{i-98}$

sln: $\sum_{i=99}^{100} \frac{2}{i-98} = \frac{2}{99-98} + \frac{2}{100-98} = \frac{2}{1} + \frac{2}{2} = \boxed{3}$.

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Formulas: ( formula sheet )

① $\sum_{i=1}^{n} c = c\cdot n$, ② $\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$, ③ $\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$

eg4. $\sum_{i=1}^{20} (-3) = -3\cdot 20 = -60$

$$\Big( = \underbrace{-3 -3 -3 - \cdots \quad -3}_{20 \text{ copies of } (-3)} \Big)$$

eg5 Find $\sum_{i=1}^{30}(3+2i)$ (Fit)

$$\sum_{i=1}^{30}(3+2i) \overset{\text{linear}}{=\!=} \sum_{i=1}^{30} 3 + \sum_{i=1}^{30} 2i$$

$$= 3\cdot 30 + 2\cdot\sum_{i=1}^{30} i$$

(Formula ②, n=30)

$$= 90 + 2\cdot\frac{30\cdot(30+1)}{2}$$

$$= 90 + 30\cdot 31 = 90+930 = \boxed{1020}$$

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Hints for webwork:

\#5. $\cos 0 = 1$, $\cos\pi = -1$, $\cos 2\pi = 1$, $\cos 3\pi = -1, \cdots$

$\cos i\pi = (-1)^i$, $i = 0, 1, 2, \cdots$

\#6. $\sum_{i=1}^{55}(-3i^2) = -3\cdot\sum_{i=1}^{55} i^2 = -3\cdot\frac{55\cdot(55+1)\cdot(2\cdot 55+1)}{6}$ (Formula)

$$\sum_{i=5}^{55}(-3i^2) = \left[\sum_{i=1}^{55}(-3i^2)\right] - \left[\sum_{i=1}^{4} -3i^2\right]$$

\#8. $\sum_{i=1}^{n}(5i^2+7i) = \sum_{i=1}^{n}5\cdot i^2 + \sum_{i=1}^{n}7i = 5\cdot\sum_{i=1}^{n}i^2 + 7\sum_{i=1}^{n}i = 5\cdot\frac{n(n+1)(2n+1)}{6} + 7\cdot\frac{n(n+1)}{2}$

\#14. $\sum_{i=1}^{50} i(7i+2) = \sum_{i=1}^{50}(7i^2+2i) = \sum_{i=1}^{50}7i^2 + \sum_{i=1}^{50}2i$ Formula ②, ③

$$= 7\cdot\frac{50\cdot(50+1)(2\cdot 50+1)}{6} + 2\cdot\frac{50(50+1)}{2}.$$

\#11. Write $4+8+12+16+20+24$ as a sigma notation

$$= 4\cdot 1 + 4\cdot 2 + 4\cdot 3 + 4\cdot 4 + 4\cdot 5 + 4\cdot 6 = \sum_{i=1}^{6} 4i$$

$$= 4(1+0) + 4(1+1) + \cdots + 4(5+1) = \sum_{i=0}^{5} 4(i+1)$$

\#12. $\sum_{i=0}^{4}(-1)^i = (-1)^0 + (-1)^1 + (-1)^2 + (-1)^3 + (-1)^4 = 1-1+1-1+1 = \boxed{1}$

# E Sigma Notation

A convenient way of writing sums uses the Greek letter $\Sigma$ (capital sigma, corresponding to our letter S) and is called **sigma notation**.

This tells us to end with $i = n$.

This tells us to add.

$$\sum_{i=m}^{n} a_i$$

This tells us to start with $i = m$.

**1 Definition** If $a_m, a_{m+1}, \ldots, a_n$ are real numbers and $m$ and $n$ are integers such that $m \leqslant n$, then

$$\sum_{i=m}^{n} a_i = a_m + a_{m+1} + a_{m+2} + \cdots + a_{n-1} + a_n$$

With function notation, Definition 1 can be written as

$$\sum_{i=m}^{n} f(i) = f(m) + f(m+1) + f(m+2) + \cdots + f(n-1) + f(n)$$

Thus the symbol $\Sigma_{i=m}^{n}$ indicates a summation in which the letter $i$ (called the **index of summation**) takes on consecutive integer values beginning with $m$ and ending with $n$, that is, $m, m+1, \ldots, n$. Other letters can also be used as the index of summation.

**EXAMPLE 1**

(a) $\displaystyle\sum_{i=1}^{4} i^2 = 1^2 + 2^2 + 3^2 + 4^2 = 30$

(b) $\displaystyle\sum_{i=3}^{n} i = 3 + 4 + 5 + \cdots + (n-1) + n$

(c) $\displaystyle\sum_{j=0}^{5} 2^j = 2^0 + 2^1 + 2^2 + 2^3 + 2^4 + 2^5 = 63$

(d) $\displaystyle\sum_{k=1}^{n} \frac{1}{k} = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}$

(e) $\displaystyle\sum_{i=1}^{3} \frac{i-1}{i^2+3} = \frac{1-1}{1^2+3} + \frac{2-1}{2^2+3} + \frac{3-1}{3^2+3} = 0 + \frac{1}{7} + \frac{1}{6} = \frac{13}{42}$

(f) $\displaystyle\sum_{i=1}^{4} 2 = 2 + 2 + 2 + 2 = 8$

**EXAMPLE 2** Write the sum $2^3 + 3^3 + \cdots + n^3$ in sigma notation.

SOLUTION There is no unique way of writing a sum in sigma notation. We could write

$$2^3 + 3^3 + \cdots + n^3 = \sum_{i=2}^{n} i^3$$

or

$$2^3 + 3^3 + \cdots + n^3 = \sum_{j=1}^{n-1} (j+1)^3$$

or

$$2^3 + 3^3 + \cdots + n^3 = \sum_{k=0}^{n-2} (k+2)^3$$

The following theorem gives three simple rules for working with sigma notation.

**2 Theorem** If $c$ is any constant (that is, it does not depend on $i$), then

(a) $\sum_{i=m}^{n} ca_i = c\sum_{i=m}^{n} a_i$  (b) $\sum_{i=m}^{n} (a_i + b_i) = \sum_{i=m}^{n} a_i + \sum_{i=m}^{n} b_i$

(c) $\sum_{i=m}^{n} (a_i - b_i) = \sum_{i=m}^{n} a_i - \sum_{i=m}^{n} b_i$

PROOF To see why these rules are true, all we have to do is write both sides in expanded form. Rule (a) is just the distributive property of real numbers:

$$ca_m + ca_{m+1} + \cdots + ca_n = c(a_m + a_{m+1} + \cdots + a_n)$$

Rule (b) follows from the associative and commutative properties:

$$(a_m + b_m) + (a_{m+1} + b_{m+1}) + \cdots + (a_n + b_n)$$
$$= (a_m + a_{m+1} + \cdots + a_n) + (b_m + b_{m+1} + \cdots + b_n)$$

Rule (c) is proved similarly.

**EXAMPLE 3** Find $\sum_{i=1}^{n} 1$.

SOLUTION
$$\sum_{i=1}^{n} 1 = \underbrace{1 + 1 + \cdots + 1}_{n \text{ terms}} = n$$

**EXAMPLE 4** Prove the formula for the sum of the first $n$ positive integers:

$$\sum_{i=1}^{n} i = 1 + 2 + 3 + \cdots + n = \frac{n(n+1)}{2}$$

SOLUTION This formula can be proved by mathematical induction (see page 76) or by the following method used by the German mathematician Karl Friedrich Gauss (1777–1855) when he was ten years old.

Write the sum $S$ twice, once in the usual order and once in reverse order:

$$S = 1 + 2 + 3 + \cdots + (n-1) + n$$
$$S = n + (n-1) + (n-2) + \cdots + 2 + 1$$

Adding all columns vertically, we get

$$2S = (n+1) + (n+1) + (n+1) + \cdots + (n+1) + (n+1)$$

On the right side there are $n$ terms, each of which is $n + 1$, so

$$2S = n(n+1) \qquad \text{or} \qquad S = \frac{n(n+1)}{2}$$

**EXAMPLE 5** Prove the formula for the sum of the squares of the first $n$ positive integers:

$$\sum_{i=1}^{n} i^2 = 1^2 + 2^2 + 3^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6}$$

SOLUTION 1 Let $S$ be the desired sum. We start with the *telescoping sum* (or collapsing sum):

Most terms cancel in pairs.

$$\sum_{i=1}^{n}[(1+i)^3 - i^3] = (\cancel{2^3} - 1^3) + (\cancel{3^3} - \cancel{2^3}) + (\cancel{4^3} - \cancel{3^3}) + \cdots + [(n+1)^3 - \cancel{n^3}]$$
$$= (n+1)^3 - 1^3 = n^3 + 3n^2 + 3n$$

On the other hand, using Theorem 2 and Examples 3 and 4, we have

$$\sum_{i=1}^{n}[(1+i)^3 - i^3] = \sum_{i=1}^{n}[3i^2 + 3i + 1] = 3\sum_{i=1}^{n} i^2 + 3\sum_{i=1}^{n} i + \sum_{i=1}^{n} 1$$
$$= 3S + 3\frac{n(n+1)}{2} + n = 3S + \tfrac{3}{2}n^2 + \tfrac{5}{2}n$$

Thus we have

$$n^3 + 3n^2 + 3n = 3S + \tfrac{3}{2}n^2 + \tfrac{5}{2}n$$

Solving this equation for $S$, we obtain

$$3S = n^3 + \tfrac{3}{2}n^2 + \tfrac{1}{2}n$$

or
$$S = \frac{2n^3 + 3n^2 + n}{6} = \frac{n(n+1)(2n+1)}{6}$$

**Principle of Mathematical Induction**
Let $S_n$ be a statement involving the positive integer $n$. Suppose that
1. $S_1$ is true.
2. If $S_k$ is true, then $S_{k+1}$ is true.

Then $S_n$ is true for all positive integers $n$.

SOLUTION 2 Let $S_n$ be the given formula.

**1.** $S_1$ is true because $$1^2 = \frac{1(1+1)(2\cdot 1+1)}{6}$$

**2.** Assume that $S_k$ is true; that is,

$$1^2 + 2^2 + 3^2 + \cdots + k^2 = \frac{k(k+1)(2k+1)}{6}$$

Then

See pages 76 and 79 for a more thorough discussion of mathematical induction.

$$1^2 + 2^2 + 3^2 + \cdots + (k+1)^2 = (1^2 + 2^2 + 3^2 + \cdots + k^2) + (k+1)^2$$
$$= \frac{k(k+1)(2k+1)}{6} + (k+1)^2$$
$$= (k+1)\frac{k(2k+1) + 6(k+1)}{6}$$
$$= (k+1)\frac{2k^2 + 7k + 6}{6}$$
$$= \frac{(k+1)(k+2)(2k+3)}{6}$$
$$= \frac{(k+1)[(k+1)+1][2(k+1)+1]}{6}$$

So $S_{k+1}$ is true.

By the Principle of Mathematical Induction, $S_n$ is true for all $n$.

We list the results of Examples 3, 4, and 5 together with a similar result for cubes (see Exercises 37–40) as Theorem 3. These formulas are needed for finding areas and evaluating integrals in Chapter 5.

**3 Theorem** Let $c$ be a constant and $n$ a positive integer. Then

(a) $\displaystyle\sum_{i=1}^{n} 1 = n$

(b) $\displaystyle\sum_{i=1}^{n} c = nc$

(c) $\displaystyle\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$

(d) $\displaystyle\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$

(e) $\displaystyle\sum_{i=1}^{n} i^3 = \left[\frac{n(n+1)}{2}\right]^2$

**EXAMPLE 6** Evaluate $\displaystyle\sum_{i=1}^{n} i(4i^2 - 3)$.

**SOLUTION** Using Theorems 2 and 3, we have

$$\begin{aligned}\sum_{i=1}^{n} i(4i^2 - 3) &= \sum_{i=1}^{n} (4i^3 - 3i) = 4\sum_{i=1}^{n} i^3 - 3\sum_{i=1}^{n} i \\ &= 4\left[\frac{n(n+1)}{2}\right]^2 - 3\,\frac{n(n+1)}{2} \\ &= \frac{n(n+1)[2n(n+1) - 3]}{2} \\ &= \frac{n(n+1)(2n^2 + 2n - 3)}{2}\end{aligned}$$

**EXAMPLE 7** Find $\displaystyle\lim_{n\to\infty} \sum_{i=1}^{n} \frac{3}{n}\left[\left(\frac{i}{n}\right)^2 + 1\right]$.

The type of calculation in Example 7 arises in Chapter 5 when we compute areas.

**SOLUTION**

$$\begin{aligned}\lim_{n\to\infty} \sum_{i=1}^{n} \frac{3}{n}\left[\left(\frac{i}{n}\right)^2 + 1\right] &= \lim_{n\to\infty} \sum_{i=1}^{n}\left[\frac{3}{n^3} i^2 + \frac{3}{n}\right] \\ &= \lim_{n\to\infty}\left[\frac{3}{n^3}\sum_{i=1}^{n} i^2 + \frac{3}{n}\sum_{i=1}^{n} 1\right] \\ &= \lim_{n\to\infty}\left[\frac{3}{n^3}\,\frac{n(n+1)(2n+1)}{6} + \frac{3}{n}\cdot n\right] \\ &= \lim_{n\to\infty}\left[\frac{1}{2}\cdot\frac{n}{n}\cdot\left(\frac{n+1}{n}\right)\left(\frac{2n+1}{n}\right) + 3\right] \\ &= \lim_{n\to\infty}\left[\frac{1}{2}\cdot 1\left(1 + \frac{1}{n}\right)\left(2 + \frac{1}{n}\right) + 3\right] \\ &= \tfrac{1}{2}\cdot 1\cdot 1\cdot 2 + 3 = 4\end{aligned}$$

# §4.2. The Definite Integral.

- The Definite Integral of $f(x)$ from $x=a$ to $x=b$ is denoted by

$$\int_a^b f(x)\,dx := \text{"Area under the curve"} = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\cdot\Delta x, \quad \Delta x = \frac{b-a}{n}, \quad x_i = a + i\cdot\frac{b-a}{n}.$$

Remark: $\int$ : integral notation  $a$: lower limit. $b$: upper limit. $dx$: integral w.r.t. $x$ variable

Remark: Integral "=" Area under the curve, only depends on $f$ and $a, b$ (is a number)

The variable $dx$ can be changed to any other variable. $\int_a^b f(x)\,dx = \int_a^b f(t)\,dt = \int_a^b f(u)\,du$

- Give a Riemann Sum, how to find the corresponding Integral $\int_a^b f(x)\,dx$?

eg.1. The limit $\lim_{n\to\infty} \sum_{i=1}^{n} \frac{1}{7+\frac{4i}{n}}\cdot\frac{4}{n}$ is the limit of a Riemann Sum for a certain definite integral $\int_a^b f(x)\,dx$. Find the exact form of $\int_a^b f(x)\,dx$.

Solution: $\frac{4}{n}$ plays the role of $\Delta x = \frac{b-a}{n}$, ie., $b-a=4$. the interval is 4 units long.

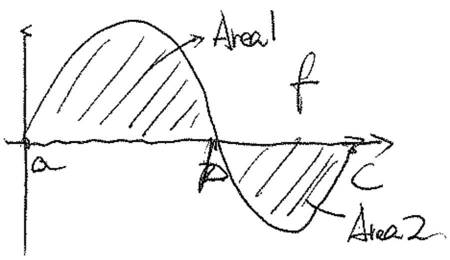
$\frac{1}{7+\frac{4i}{n}}$ plays the role of $f(x_i)$, which suggests $f$ is $\frac{1}{\cancel{\quad}}$

Let $f(x) = \frac{1}{x}$.  And our $x_1 = 7+\frac{4}{n}$, $x_n = 7+4$, $x_i = 7+i\cdot\frac{4}{n}$.

Pick $a=7$, $b=11$. Then $\int_7^{11} \frac{1}{x}\,dx$ fits the Riemann Sum.

Remark: The answer is not unique. One can check $\int_0^4 \frac{1}{7+x}\,dx$ is also correct.

- The integral represents "the area with signs". If the curve is under the x-axis, the area is considered as negative.



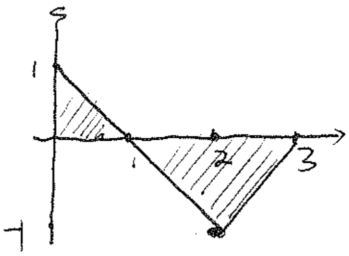

$\int_a^b f(x)\,dx = \text{Area}1$.  $\int_b^c f(x)\,dx = -\text{Area}2$

$\int_a^c f(x)\,dx = \text{Area}1 - \text{Area}2$.

eg.2. Consider the function $f(x)=\begin{cases} 1-x & 0\le x\le 2 \\ x-3 & 2\le x\le 3 \end{cases}$

Use the graph of $f(x)$ on $[0,3]$ to find $\int_0^1 f(x)dx$, $\int_1^2 f(x)dx$, $\int_0^2 f(x)dx$, $\int_1^3 f(x)dx$, $\int_0^3 f(x)dx$

Solution:

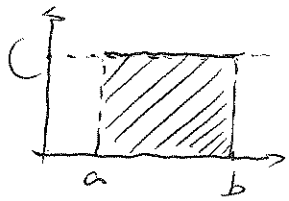


$\int_0^1 f(x)dx = \frac{1}{2}\cdot 1\cdot 1 = \frac{1}{2}$, $\int_1^2 f(x)dx = -\frac{1}{2}$, $\int_1^3 f(x)dx = -1$

$\int_0^2 f(x)dx = \frac{1}{2}-\frac{1}{2} = 0$. $\int_0^3 f(x)dx = \frac{1}{2}-1 = -\frac{1}{2}$.

- Some basic integrals from the graph:

Rectangle $\int_a^b C\,dx = C(b-a)$.

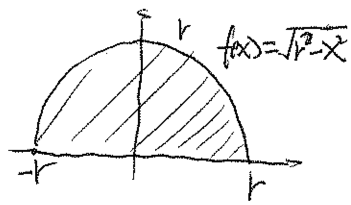


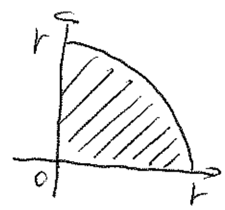
$f(x) = C$ (constant C)

Area $= C(b-a)$.

Half/Quarter Disk: $\int_{-r}^{r} \sqrt{r^2-x^2}\,dx = \frac{1}{2}\pi r^2$

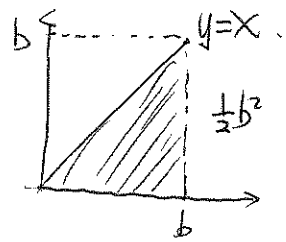
$\int_0^r \sqrt{r^2-x^2}\,dx = \frac{1}{4}\pi r^2$

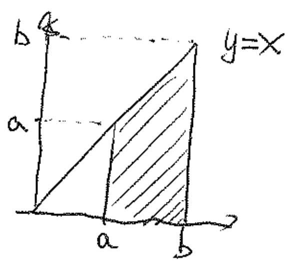


Triangle/Trapezoid: $\int_0^b x\,dx = \frac{1}{2}b^2$

$\int_a^b x\,dx = \frac{1}{2}b^2-\frac{1}{2}a^2$

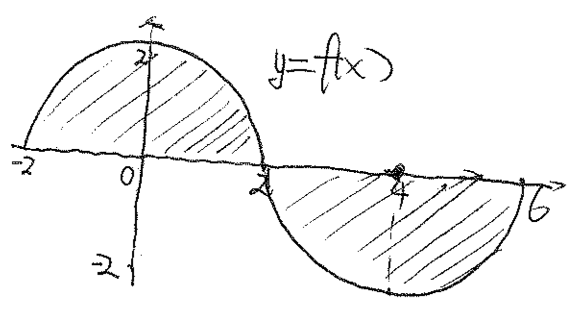



eg.3. Suppose the graph of $y=f(x)$ is as follow.

All curves are half circles with radius 2.

Find. $\int_{-2}^{2} f(x)dx$, $\int_0^2 f(x)dx$, $\int_0^4 f(x)dx$




$\int_{-2}^{2} f(x)dx = \frac{1}{2}\pi\cdot 2^2 = 2\pi$

$\int_0^2 f(x)dx = \frac{1}{4}\pi\cdot 2^2 = \pi$

$\int_0^4 f(x)dx = \frac{1}{4}\pi\cdot 2^2 - \frac{1}{4}\pi\cdot 2^2 = 0$

$\int_2^6 f(x)dx = -\frac{1}{2}\pi\cdot 2^2 = -2\pi$

$\int_0^6 f(x)dx = \pi-(2\pi) = -\pi$. $\int_{-2}^{6} f(x)dx = 2\pi - 2\pi = 0$

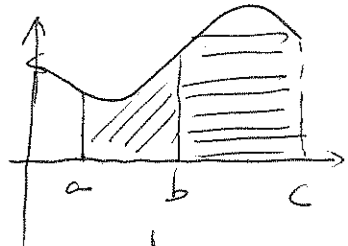
- Properties of Definite integrals

① $\int_a^a f(x)\,dx = 0$. (The area is zero if the upper and lower limits coincide).

② $\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$. (Flip the lower and upper limits by adding a negative sign)

eg. $\int_5^2 2x\,dx = -\int_2^5 2x\,dx$.

☆ ③ Splitting $\int_a^c f(x)dx = \int_a^b f(x)dx + \int_b^c f(x)dx$.




④ linear properties $\int_a^b [f(x) \pm g(x)]\,dx = \int_a^b f(x)dx \pm \int_a^b g(x)dx$ ; $\int_a^b C f(x)dx = C \int_a^b f(x)dx$.

☆ eg4. (#16) suppose $\int_2^5 f(x)\,dx = 3$, $\int_2^3 f(x)\,dx = -4$. Find $\int_5^3 2f(x)\,dx$.

Solution: $\int_5^3 2f(x)\,dx = \int_5^2 2f(x)\,dx + \int_2^3 2f(x)\,dx$ splitting $\int_5^3 = \int_5^2 + \int_2^3$

$= -\int_2^5 2f(x)dx + \int_2^3 2f(x)dx$. flipping $\int_5^2 = -\int_2^5$

$= -2\cdot\int_2^5 f(x)dx + 2\cdot\int_2^3 f(x)dx$. constant multiple

$= -2\cdot 3 + 2\cdot(-4) = \boxed{-14}$.

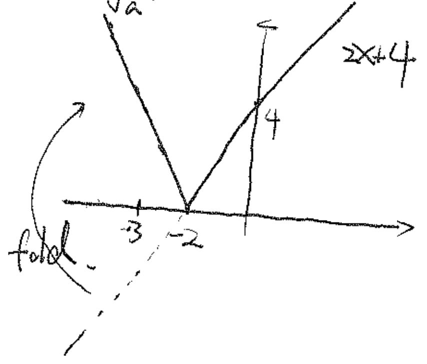
---

Hints for webwork:

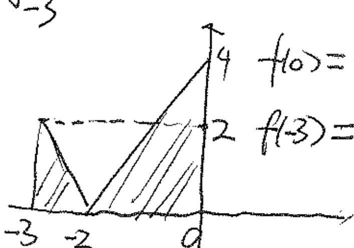
※10: ⑤ Bounds: If $A \le f(x) \le B$, then $A(b-a) \le \int_a^b f(x)dx \le B(b-a)$

☆ ※5. Graph of abstract value:

$y = |2x+4|$



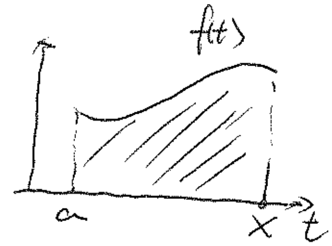

$\int_{-3}^{0} |2x+4|\,dx = \frac{1}{2}\cdot 1\cdot 2 + \frac{1}{2}\times 2\times 4 = 5$




$f(0) = 4$

$f(-3) = |2\cdot(-3)+4| = 2$

# § Fundamental Theorem of Calculus.




Key formula: ① FTC P1: If $F(x)=\int_a^x f(t)\,dt$, then $F'(x)=f(x)$.

★ If $F(x)=\int_{V(x)}^{U(x)} f(t)\,dt$, then

$$F'(x)=f(U(x))\cdot U'(x)-f(V(x))\cdot V'(x). \quad \text{(chain rule form)}$$

$$\left(\int_a^{U(x)} f(t)\,dt\right)'=f(U(x))\cdot U'(x), \quad \left(\int_{V(x)}^b f(t)\,dt\right)'=-f(V(x))\cdot V'(x).$$

② FTC P2: If $F'(x)=f(x)$ ($F$ is one anti-D of $f(x)$)

then $\int_a^b f(x)\,dx = F(x)\Big|_a^b = F(b)-F(a)$.

③ Anti D (Integral Table):

$(n\neq -1)$: $\int_a^b x^n\,dx=\frac{1}{n+1}x^{n+1}\Big|_a^b$, $\int_a^b \cos x\,dx=\sin x\Big|_a^b$, $\int_a^b \sin x\,dx=-\cos x\Big|_a^b$

$\int_a^b \sec^2 x\,dx=\tan x\Big|_a^b$, $\int_a^b \sec x\cdot\tan x\,dx=\sec x\Big|_a^b$.

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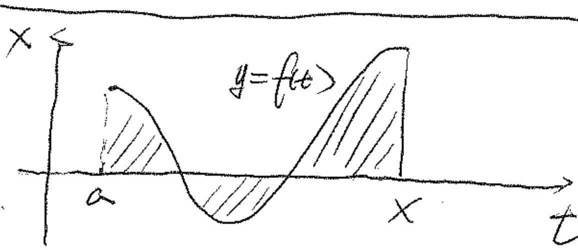
Consider $y=f(t)$. from $t=a$ to $t=x$.

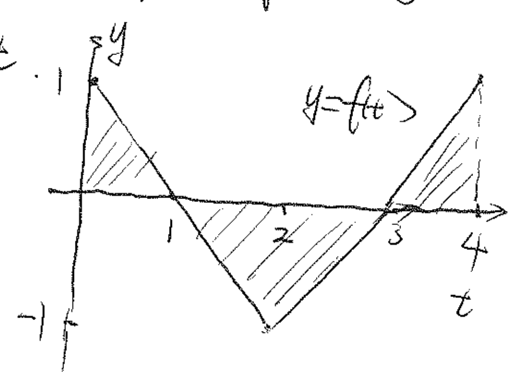



$\int_a^x f(t)\,dt$ is the area of the shadow region.

It changes as $x$ moves, therefore, is a function of the upper limit $x$.

We denote it by $F(x)=\int_a^x f(t)\,dt$. FTC P1, P2 tell us the relation between $f$ and $F$ and how to use anti D to compute a definite integral.

e.g. 1. $y=f(t)$ the graph is given below. let $g(x)=\int_0^x f(t)\,dt$.




$g(0)=\int_0^0 f(t)\,dt=0$, $g(1)=\int_0^1 f(t)\,dt=\frac{1}{2}$, $g(2)=0$

$g(3)=-\frac{1}{2}$, $g(4)=0$

★ $g(x)$ is increasing on $[0,1]\cup[3,4]$. decreasing on $[1,3]$.

- Derivative formulas:

e.g.2. Find $\frac{d}{dx}\int_{-9}^{x}(\cos t^2 + t)\,dt = \left(\int_{-9}^{x}\cos t^2 + t\;dt\right)'$

sln: Apply FTC P1 with $f(t) = \cos t^2 + t$. (Replace $t$ in $f(t)$ by $x$).

$$\left(\int_{-9}^{x}\cos t^2 + t\;dt\right)' = \cos x^2 + x.$$

eg3. Let $h(x) = \int_{\tan x}^{3}\sqrt{2+t^2}\,dt$. Find $h'(x)$.

sln: Apply FTC P1 with $u(x) = 3$ (const), $V(x) = \tan x$. $f(t) = \sqrt{2+t^2}$

$$h'(x) = 0 - \sqrt{2+(\tan x)^2}\cdot(\tan x)' = -\sqrt{2+(\tan x)^2}\cdot\sec^2 x.$$

↑ replace $t$ by lower limit $\tan x$.

- Evalute the definite integral by finding the anti-D

eg.4. Evaluate $\int_0^{\frac{\pi}{3}} 4\cdot\sec x\cdot\tan x\,dx$.

Solution: $\int_0^{\frac{\pi}{3}} \underline{4\cdot\sec x\cdot\tan x}\,dx$. Step 1: $f(x) = 4\sec x\,\tan x$. Find anti-D $F(x)$ of $f(x)$

$F(x) = 4\cdot\sec x$. (since $(\sec x)' = \sec x\cdot\tan x$)

Step2: FTC P2: $\int_0^{\frac{\pi}{3}} 4\sec x\cdot\tan x\,dx = 4\sec x\Big|_0^{\frac{\pi}{3}} = 4\sec\frac{\pi}{3} - 4\sec 0$. Hint: $\sec\frac{\pi}{3} = \frac{1}{\cos\frac{\pi}{3}} = 2$

$= 4\cdot 2 - 4\cdot 1 = \boxed{4}$ $\qquad \sec 0 = \frac{1}{\cos 0} = 1$

eg.5 Evaluate $\int_1^{\sqrt{7}} \frac{13\cdot s^4 + 5\cdot\sqrt{s}}{s^4}\,ds$

Solution: $f(s) = \frac{13\cdot s^4 + 5\cdot\sqrt{s}}{s^4} = \frac{13\cdot s^4}{s^4} + \frac{5\cdot s^{\frac{1}{2}}}{s^4} = 13 + 5\cdot s^{\frac{1}{2}-4} = 13 + 5\cdot s^{-\frac{7}{2}}$

anti-D: $F(s) = 13\cdot s + 5\cdot\frac{1}{-\frac{7}{2}+1}\cdot s^{-\frac{7}{2}+1} = 13s + 5\cdot\frac{1}{-\frac{5}{2}}\cdot s^{-\frac{5}{2}}$

$= 13s - 2\cdot s^{-\frac{5}{2}}$

FTC P2 $\Rightarrow \int_1^{\sqrt{7}} \frac{13s^4 + 5\sqrt{s}}{s^4}\,ds = (13\cdot s - 2\cdot s^{-\frac{5}{2}})\Big|_1^{\sqrt{7}} = \left(13\cdot\sqrt{7} - 2(\sqrt{7})^{-\frac{5}{2}}\right) - \left(13\cdot 1 - 2\cdot 1^{-\frac{5}{2}}\right)$

$= \boxed{13\cdot\sqrt{7} - 2\cdot(\sqrt{7})^{-\frac{5}{2}} - 11}$